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ISSN 0304-9892

Jñānābha ज्ञानाभा

12

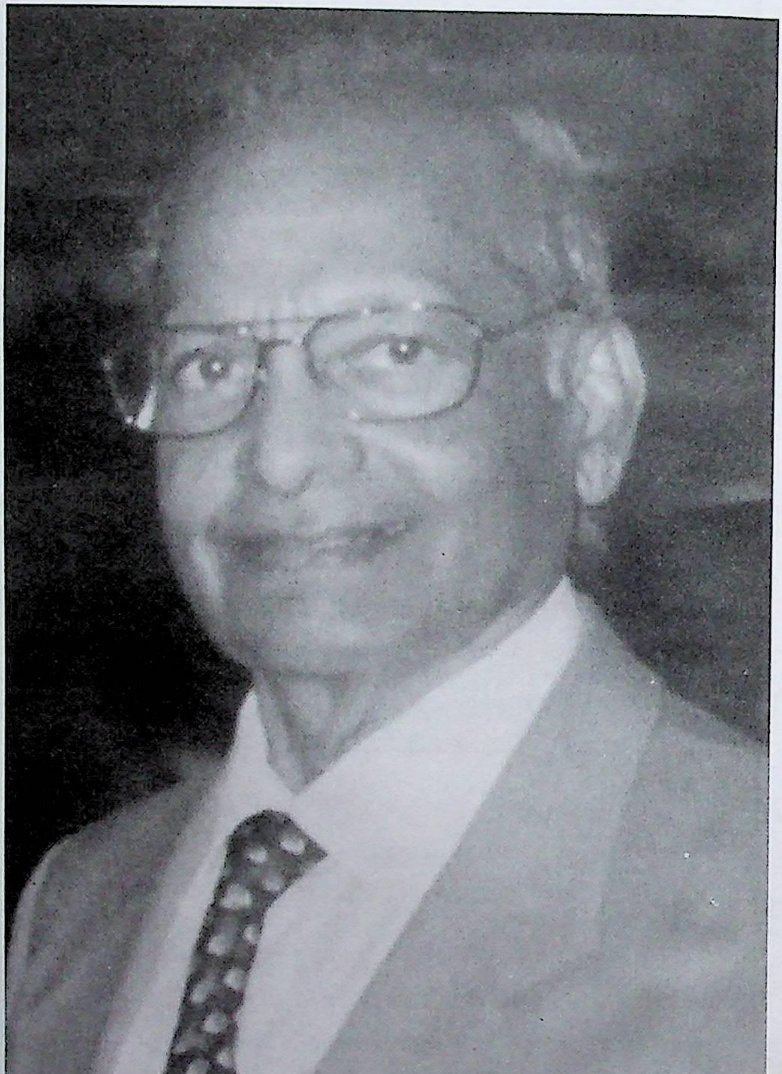
VOLUME 45

Special Volume to Honour
Professor H.M. Srivastava
During his Platinum Jubilee Celebrations

2015

Published by :
The Vijñāna Parishad of India
DAYANAND VEDIC POSTGRADUATE COLLEGE
(Bundelkhand University)
ORAI-285001, U.P., INDIA
www.vijnanaparishadofindia.org/jnanabha

This Special Volume of
JÑĀNĀBHA
is being dedicated to honor
Professor H.M. Srivastava
(During his Platinum Jubilee Celebration)



PROFESSOR H.M. SRIVASTAVA

(Born : July 05, 1940)

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

PROFESSOR H.M. SRIVASTAVA : MAN AND MATHEMATICIAN

By

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We feel great honored to honor Professor H.M. Srivastava during his Platinum Jubilee Celebrations. He Needs no introduction. He is an amazing man, towering and leading mathematician, well known topmost eminent figure of "Special Functions and Allied Topics of Mathematical Analysis". He is topmost Researcher, Supervisor of several Ph.D. and D.Sc. theses, well reputed University Teacher, Editor or Member on Editorial Boards of various International Research Journals, Reviewer of various Reviews, Elected Follows of various International Societies, Recipient of International Prizes, Awards, Honors, Author of various Internationally prescribed Text Books having Special Dedication Volumes/Dedication Issues (and/or Dedication messages) of International Scientific Research Journals.

Professor H.M. Srivastrava is well known to me since 1967 when he was faculty member of Jodhpur University/West Verginia University USA and I was a regular Research Scholar (Ministry of Education Government of India) working in his field at SATI, Vidisha, M.P., India under his postal guidance along with my guide Proferssor P.M. Gupta (Vidhisha). Due to this fact Professor Srivastava is well associated with JÑĀNĀBHA family since very inception *Jñānābha* , Vol. **1** (1971), when he was active member on its Editorial Board. Since *Jñānābha* , Vol.**2**, 1972, Professor Srivastava is working as one of the two Editors along with Foreign Secretary of Vijñāna Parishad of India, which publishes *Jñānābha* . To bring Jñānābha and *Vijñāna Parishad of India* on present World map, Professor Srivastava has an important role.

Professor Srivastava was elected as one of the first two Honorary Fellows of Vijñāna Prishad of India (F.V.P.I.) (with first Fellow Late Professor J.N. Kapur) during Silver Jubilee Conference of the Parishad

held at Parishad Head Quarters : D.V. Postgraduate, Orai, Uttar Pradesh, India in May 1996.

Special Issue *Jñānābha* Vol. **31/32**, 2002 (Dedicated to Honour Professor H.M. Srivastava on his 62^{nd} Birthday) was also published by **Vijñāna Parishad of India.**

Professor Srivastava, the ocean of generosity has inspired and helped several students, teachers and colleagues to accept multi-dimensional challenges of physical real life.

Thus for his outstanding contribution and Distinguished Services rendered to his Subject, " Vijñāna Parishad of India" and to Social Communities, **President, Members and Fellows of Vijñāna Parishad of India** feel great pleasure to **honor Professor H.M Srivastava**

1. by publishing Special Issue of *Jñānābha* Vol. **45**, 2005 (Dedicated to Honor Professor H.M. Srivastava During his Platinum Jubilee Celibrations)

and

2. by organizing **18^{th} Cum 1^{st} International Conference of Vijñāna Parishad of India** (*Dedicated to Honor Professor H.M. Srivastava During His Platinum Jubilee Celebrations) at MANIT, Bhopal, M.P., India (December* 11-14, 2015*)*.

I. A Brief Biographical Sketch. Professor Hari Mohan Srivastava was born on 5th July 1940 at Karon in District Ballia of the Province of Uttar Pradesh in India. His father, Sri Harihar Prasad (1900-1985), was a lawyer practising in the Civil and District Courts at Ballia. His mother, Srimati Bela Devi (1910-1989), was remarkably well-versed in the Vedic and Hindu religious scriptures which greatly influenced his childhood and later life spiritually as well as culturally. His father, on the other hand, signicantly strengthened his pre-university education especially in the subjects of English and Mathematics.

Having had hardly any formal education at the Primary School level, Professor Srivastava was admitted in 1946 directly into Grade 3 of the Government Higher Secondary School at Ballia at the age of 6 years after his successful performance in the mandatory written and oral entrance examinations. It is at this School where he was awarded a double

promotion from Grade 4 to Grade 6, without having to go through Grade 5. This did indeed accelerate his completion of High School (Grade 10) in 1953. During the next two years (1953-1955), he studied at B.N.V. College at Rath in District Hamirpur of the Province of Uttar Pradesh, where he completed his I. Sc. in 1955, breaking all existing academic and scholarly records of that College as well as in other colleges in the region.

Professor Srivastava received his university education at the University of Allahabad where he completed his B. Sc. in 1957 and M. Sc. in 1959. Besides being a throughout high First class [right from High School (1953) to M. Sc. (1959)] and meritorious product of the University of Allahabad, and having won a number of merit prizes and scholarships, he was awarded the *Allahabad Jubilee Medal* in the year 1959. During the period (1955-1959) of his four-year stay at the University of Allahabad, Professor Srivastava also published two first-prize-winning short stories in English, which were subsequently translated and published in other Indian languages (especially in Hindi).

Professor Srivastava began his university-level teaching career in 1959 itself at the age of 19 years. He taught at D. M. Government College in Imphal (now Manipur University) during the academic year 1959-1960 and at the University of Roorkee (now the Indian Institute of Technology at Roorkee) during the academic years 1960-1963. He then moved to Jodhpur University (now Jai Narain Vyas University) where he earned his Ph. D. degree in 1965 while he was a full-time member of the teaching faculty at Jodhpur University (since 1963).

Currently, Professor Srivastava holds the position of Professor Emeritus in the Department of Mathematics and Statistics at the University of Victoria in Canada. He joined the faculty there in 1969 [first as Associate Professor (1969-1974) and then as Full Professor (1974-2006)]. Professor Srivastava has held numerous visiting positions including (for example) those at West Virginia University in U.S.A. (1967-1969), Université Laval in Canada (1975), and the University of Glasgow in U.K. (1975-1976), and indeed also at many other universities and research institutes in dierent parts of the world. He is also Honorary Advisory Professor and Honorary Chair Professor at many universities.

Professor Srivastava's academic as well as personal life has been greatly enriched by the dedicated and whole-hearted support of his wife, Prof. Dr. Rekha Srivastava, who is also a mathematician and colleague in the same Department of Mathematics and Statistics at the University of Victoria, and by his two children, Sapna Srivastava (who is currently working as a Journalist in the New York area in U.S.A. after her Master's degree in Journalism from Fordham University in New York) and Dr. Gautam Mohan Srivastava (who is currently teaching in the Department of Mathematics and Computer Science at Brandon University in Canada). Many of Professor Srivastava's teachers (especially those at the University of Allahabad), too, deserve to be credited for his choice of a teaching career and for his academic and scholarly accomplishments in his chosen profession.

When not fully immersed into his research and writing, Professor Srivastava prefers to pursue one of his main hobbies: watching movies and serials (mostly in Hindi) on largescreen television at home with his family. His continuing interest in sports is exemplified by his active participation in hockey games until recently **and** by his regular attendance at baseball games - live (especially when his son, who is presently **also** a successful (**and nationally well-recognized**) Baseball Coach, used to play) or on television-with his wife (who incidentally got him deeply interested in baseball games, too). Besides, the spiritual and religious inclinations of Professor Srivastava and his wife, which were implanted in them by the cultural and spiritual environment of their respective families, grew much stronger in their own family life. He and his wife, together with their children, have contributed signicantly to community and other related services to the society.

II. Honours, Awards and Other Accomplishments. Professor Srivastava has published 23 books, monographs and edited volumes, 31 book (and encyclopedia) chapters, 45 papers in international conference proceedings, and **over 1,000** scientific research journal articles on various topics of mathematical analysis and applicable mathematics. In addition, he has written Forewords and Prefaces to several books by other authors and to several special issues of scientific journals. He has also edited (and contributed to) many volumes which are dedicated to the memories of

famous mathematical scientists. Citations of his research contributions can be found in many books and monographs, Ph.D. and D.Sc. theses, and scientific journal articles, much too numerous to be recorded here. Currently, he is actively associated editorially (that is, as an Editor-in-Chief, Editor, Honorary Editor, Advisory Editor, Senior Editor, Associate Editor or Editorial Board Member) with **over 200** international scientific research journals. His biographical sketches (many of which are illustrated with his photograph) have appeared in various issues of more than 50 international biographies, directories, and **Who's Who's.**

Professor Srivastava's over 55-year career as a university-level teacher and as a remarkably prolic researcher in many dierent areas of the mathematical, physical, and statistical sciences is highlighted by (among other things) the fact that he has collaborated and published joint papers with as many as **500** mathematicians, physicists, statisticians, chemists, astrophysicists, geochemists, as well as information and business management scientists, who are scattered throughout the world, thereby qualifying for his **Erdös number** 2, implying that at least one of Professor Srivastava's co-authors is a co-author of the famous Hungarian mathematician, Paul Erdös (1913-1996). Professor Srivastava's collaboration distances with other famous scientists include his **Einstein number** 3, **Pólya number** 3, von **Neumann number** 3, **Wiles number** 3, and so on.

In the **leading** newspaper, **The Globe** and **Mail** (Toronto, March 27, 2012, Page B7 et seq.), Professor Srivastava was listed in the **Second Place** among **Canada's Top Researchers** in the discipline of **Mathematics and Statistics** in Terms of **Productivity** and **Impact Based Upon a Measure of Citations** of **Their Published Works**.

Some of the **most recent** prizes and distinctions awarded to Professor Srivastava include (for example) the following items:

1. **NSERC 25-Year Award:** University of Victoria, Canada (2004)
2. **The Nishiwaki Prize:** Japan (2004)
3. **Doctor of Science (Honoris Causa)**: Chung Yuan Christian University, Chung-Li, Taiwan, Republic of China (2006)
4. **Doctor of Science (Honoris Causa):** "1 Decembrie 1918" University of Alba Iulia, Romania (2007)

5. **Thomson-Reuters Highly Cited Researcher** (2015)
6. **Special Dedication Volumes and Special Dedication Issues of (and/or Dedication Messages in) International Scientic Research Journals**:

Fractional Calculus and Applied Analysis, Volume **3**, Number 3, 2000 (see Pages 319 {325) (Dedication Message for his 60th Birth Anniversary).

Jñānābha, Volume **31/32**, 2002 (Special Issue Dedicated to his 62nd Birthday).

Fractional Calculus and Applied Analysis, Volume **8**, Number 4, 2005 (see Pages 387-392) (Dedication Message for his 65th Birth Anniversary).

Applied Mathematics and Computation, Volume **187**, Number 1, 2007 (Special Issue Dedicated to his 65th Birth Anniversary).

Bulletin of Mathematical Analysis and Applications, Volume **4**, Number 2, 2010 (Special Issue Dedicated to his 70th Birth Anniversary).

Fractional Calculus and Applied Analysis, Volume **13**, Number 3, 2010 (see Page 342) (Dedication Message for his 70th Birth Anniversary).

European Journal of Pure and Applied Mathematics (Special Issue on Complex Analysis: Theory and Applications), Volume **3**, Number 6, 2010 (Special Issue Dedicated to his 70th Birthday).

Fractional Calculus and Applied Analysis, Volume **13**, Number 4, 2010 (Special Issue Dedicated to his 70th Birth Anniversary).

Applied Mathematics and Computation, Volume **218**, Number 3, 2011 (Special Issue Dedicated to his 70th Birth Anniversary).

Advances in Difference Equations (Springer Open-Access Journal), Volume **2013**, 2013 (Special Issue: Proceedings of the International Congress in Honour of Professor Hari M. Srivastava).

Boundary Value Problems (Springer Open-Access Journal), Volume **2013**, 2013 (Special Issue: Proceedings of the International Congress in Honour of Professor Hari M. Srivastava).

Fixed Point Theory and Applications (Springer Open-Access Journal), Volume **2013**, 2013 (Special Issue: Proceedings of the International Congress in Honour of Professor Hari M. Srivastava).

Journal of Inequalities and Applications (Springer Open-Access Journal), Volume **2013**, 2013 (Special Issue: Proceedings of the International Congress in Honour of Professor Hari M.Srivastava).

Analytic Number Theory, Approximation Theory, and Special Functions: In Honor of Hari M. Srivastava (xi+880 pp.; ISBN 978-1-4939-0257-1) (Gradimir V. Milovanovic and Michael Th. Rassias, Editors), Springer, Berlin, Heidelberg and New York, 2014.

III. Research Contributions. Many mathematical entities and objects are attributed to (and named after) him. These entities and objects include (among other items) Srivastava's polynomials and functions, Carlitz -Srivastava polynomials, Srivastava-Buschman polynomials, Srivastava-Singhal polynomials, Chan-Chyan-Srivastava polynomials, Erkus-Srivastava polynomials, Srivastava-Daoust multivariable hypergeometric function, Srivastava-Panda multivariable H-function, Singhal Srivastava generating function, Srivastava-Agarwal basic (or q-) generating function, and Wu-Srivastava inequality in the field of Higher Transcendental Functions; Srivastava-Owa, Choi-Saigo-Srivastava, Jung-Kim-Srivastava, Liu-Srivastava, Cho-Kwon-Srivastava, Dziok-Srivastava, Srivastava-Attiya, Srivastava-Wright and Srivastava-Gaboury operators in the field of Geometric Function Theory in Complex Analysis; Srivastava-Gupta operator in the field of Approximation Theory; Srivastava, Adamchik-Srivastava and Choi-Srivastava constants and methods in the field of Analytic Number Theory; and so on.

Professor Srivastava has supervised (and is currently supervising) a number of postgraduate students working toward their Master's, Ph.D. and/or D.c. degrees in different parts of the world. Besides, many post-doctoral fellows and research associates have worked with him at West Virginia University in U. S. A. and at the University of Victoria in Canada.

Some of the significant and remarkable contributions by Professor Srivastava are being listed below under each of the **main** topics of his **current** research interests:

(i) Real and Complex Analysis: A unified theory of numerous potentially useful function classes, and of various integral and convolution operators using hypergeometric functions, especially in Geometric Function Theory in Complex Analysis, and several classes of analytic and geometric inequalities in the field of Real Analysis.

(ii) Fractional Calculus and Its Applications: Generalizations of such classical fractional calculus operators as the Riemann-Liouville and Weyl

operators together with their fruitful applications to numerous families of differential, integral, and integro-differential equations, especially some general classes of fractional kinetic equations and also to some Volterra type integro-differential equations which emerge from the unsaturated behavior of the free electron laser.

(iii) Integral Equations and Transforms: Explicit solutions of several general families of dual series and integral equations occurring in Potential Theory; United theory of many known generalizations of the classical Laplace transform (such as the Meijer and Varma transforms) and of other multiple integral transforms by means of the Whittaker $W_{k,\mu}$- function and the (Srivastava-Panda) multivariable H-function in their kernels.

(iv) Higher Transcendental Functions and Their Applications: Discovery, introduction, and systematic (and united) investigation of a set of 205 triple Gaussian hypergeometric series, especially the triple hypergeometric functions H_A, H_B and H_C added to the 14-member set conjectured and defined in 1893 by Giuseppe Lauricella (1867-1913). United theory and applications of the multivariable extensions of the celebrated higher transcendental (ψ-and H-) functions of Charles Fox (1897-1977) and Edward Maitland Wright (1906-2005), and also of the Mittag-Leffler E-functions which are named after Gustav Mittag-Leffler (1846-1927). Mention should be made also of his applications of some of these Higher Transcendental Functions in Quantum and Fluid Mechanics, Astrophysics, Probability Distribution Theory, Queuing Theory and other related Stochastic Processes, and so on.

(v) q-Series and q-Polynomials: Basic theory of general q-polynomial expansions for functions of several complex variables, extensions of several celebrated q-identities of Srinivasa Ramanujan (1887-1920), and systematic introduction and investigation of multivariable basic (or q-) hypergeometric series.

(vi) Analytic Number Theory. Presentation of several computationally-friendly and rapidly-converging series representations for Riemann's Zeta function, Dirichlet's L-series, introduction and application of some novel techniques for closed-form evaluations of series involving a wide variety of sequences and functions of analytic number theory, and so on. His applications of (especially) the Hurwitz-Lerch Zeta function in Geometric

Function Theory in Complex Analysis and in Probability Distribution Theory and related topics of Statistical Sciences deserve to be recorded here.

(vii) Analytic and Geometric Inequalities. United presentations and generalizations of a number of analytic and geometric inequalities.

(viii) Probability and Statistics. Probabilistic derivations of generating functions and statistical applications of various special functions and orthogonal polynomials.

(ix) Inventory Modelling and Optimization. Systematic analytical investigation of many potentially useful problems in supply chain management.

Professor Srivastava's publications have been reviewed by (among others) **Mathematical Reviews** (U.S.A.), **Referativnyi Zhurnal Matematika (Russia), Zentralblatt foür Mathematik** (Germany), and **Applied Mechanics Reviews** (U.S.A.) under various **2010 Mathematical Subject Classifications (Math Sci Net)** including (for example) the following general classifications:

01 History and Biography
05 Combinatorics
11 Number Theory
15 Linear and Multilinear Algebra; Matrix Theory
26 Real Functions
30 Functions of a Complex Variable
31 Potential Theory
33 Special Functions
34 Ordinary Differential Equations
35 Partial Differential Equations
39 Difference and Functional Equations
40 Sequences, Series, Summability
41 Approximations and Expansions
42 Fourier Analysis
44 Integral Transforms, Operational Calculus
45 Integral Equations
46 Functional Analysis
47 Operator Theory

58 General Global Analysis, Analysis on Manifolds
60 Probability Theory and Stochastic Processes
62 Statistics
76 Fluid Mechanics
78 Optics, Electromagnetic Theory
81 Quantum Theory
85 Astronomy and Astrophysics
90 Operations Research, Mathematical Programming
91 Game Theory, Economics, Social and Behavioral Sciences
93 Systems Theory, Control

Further details about Professor Srivastava can be found at the following Web Site which is maintained and regularly updated by the University of Victoria: Web Site: http://www.math.uvic.ca/faculty/harimsri/

ACKNOWLEGEMENTS

We wish to record our deepest and sincerest feelings of gratitude to Professor Rekha Srivastava (University of Victoria, B.C., Canada) the wife of Professor H.M. Srivastava, for her kind help to collect the data cited above.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

VON MISES DISTRIBUTION : PROPERTIES AND CHARACTERIZATIONS

By

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(*Received : July 10, 2015; Final From: September 15, 2015*)

ABSTRACT

The von Mises distribution is one of the most important distributions in statitics to deal with the circular or directional data. In this paper, motivated by the importance of the von Mises distribution in many practical problems in circular or directional data, we will consider its several distributional properties. Based on these distributional properties, we have established some new characterizations of the von Mises distribution by truncated first moment, order statistics and upper record values. It is hoped that the findings of the paper will be useful for researchers in the fields of probability, statistics, and other applied sciences.

2010 Mathematics Subject Classifications : 60E05, 62E10, 62E15, 62G30.

Keywords : Characterization, von Mises distribution, Order statistics, Reverse hazard rate, Truncated first moment, Upper record values.

1. Introduction. An angular random variable X is said to have the general von Mises distribution, denoted as $X \sim vM(\theta, \mu, k)$, if its probability density functions (*pdf*), $f(\theta; \mu, k)$ is given by

$$f(\theta;\mu,k) = \frac{e^{k\cos(\theta-\mu)}}{2\pi I_0(k)},\ \mu-\pi<\theta<\mu+\pi, k\geq 0, -\infty<\mu<\infty, \tag{1.1}$$

where θ is measured in radian, k is the scale parameter (a concentration parameter), μ is the location parameter (a mean direction), and $I_0(k)$ is the modified Bessel function of the first kind and order 0, defined as follows, see, for example, Abramowitz and Stegun ([1], 1970), Gradshteyn and Ryzhik ([10],1980), and Oldham et al. ([19],2009):

$$I_0(k) = \sum_{i=0}^{\infty}\left(\frac{k}{2}\right)^{2i}\left(\frac{1}{i!}\right)^2 .$$

The mean direction coincides with the mode, whereas the parameter k measures the concentration around the mean directions μ in such a way that, as k increases, the density peaks higher around μ.

The von Mises distribution is one of the most important distributions in statistics to deal with the circular or directional data, and is the circular analog of the normal distribution on a line. The distribution named after Von Mises was introduced as a statistical model by von Mises ([22], 1918) to study deviations of atomic weights from integer values. This distribution was introduced earlier by Langevin ([14], 1905), adn Gumbel et al. ([11],1953) studied this distribution and called it circular normal distribution. Since then von Mises distribution has become very important in the statistical theory of directional data, and has been applied in areas such as geology, wildlife movement, bird navigation, among others. For details on these, the interested readers are referred to Mardia ([15],1972, [16], 1975), Kendall ([13], 1974), Batschelet ([5],1981), Fisher et al. ([9],1987), Fisher ([8],1993), Patel and Read ([20] , 1996), Mardia and Jupp ([17], 1999), Jammalamadaka and Sen-Gupta ([12], 2001), among others. In this paper, motivated by the importance of the von Mises distribution in many practical problems in circular or directional data, we will consider its several distributional properties. Based on these distributional properties, we have established some new characterizations of the von Mises distribution by truncated first moment, order statistics and upper record values. The organization of the paper is as follows. For the sake of completeness, some properties of the von Mises distribution are presented in Section 2. The characterizations of the von Mises distribution are

presented in Section 3. The concluding remarks are provided in Section 4. We have provided two lemmas in Appendix A to prove the main results of the paper.

2. Some Distributional Properties of Von Mises Distribution. In this section, we briefly discuss some of the distributional properties of the von Mises distribution. Without loss of generality μ will be taken as zero in (1.1). Then, we say that the random variable X has the standard von Mises distribution, denoted as $X \sim vM(\theta,k)$, and its *pdf* is given by

$$f(\theta)=\frac{e^{k\cos\theta}}{2\pi I_0(k)}, -\pi<\theta<\pi, k\geq 0, \tag{2.1}$$

where $I_0(k)$ is the modified Bessel function of the first kind and order 0. Since

$$e^{k\cos\theta}=\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos j\theta\right),$$

see Abramowitz and Stegun (1970), Eq. 9.6.34, page 376, we can wite the *pdf* $f(\theta)$ and the *cdf* $F(\theta)$ as follows

$$f(\theta)=\frac{1}{2\pi}\left(1+\frac{2}{I_0(k)}\sum_{j=1}^{\infty}I_j(k)\cos j\theta\right), \qquad -\pi<\theta<\pi, k\geq 0, \tag{2.2}$$

and

$$F(\theta)=\frac{1}{2\pi}\left((\theta+\pi)+\frac{2}{I_0(k)}\sum_{j=1}^{\infty}\frac{I_j(k)}{j}\sin j\theta\right), \quad -\pi<\theta<\pi, k\geq 0, \tag{2.3}$$

where $I_j(k)$ is the modified Bessel function of the first kind and order j given by

$$I_j(k)=\left(\frac{k}{2}\right)^j\sum_{i=0}^{\infty}\left(\frac{k}{2}\right)^{2i}\frac{1}{i!\Gamma(j+1+1)}.$$

To describe the shapes of the standard von Mises distribution, $X \sim vM(\theta,k)$, the plots of the pdf (2.1) for some values of the parameter k are provided below in Figure 2.1.

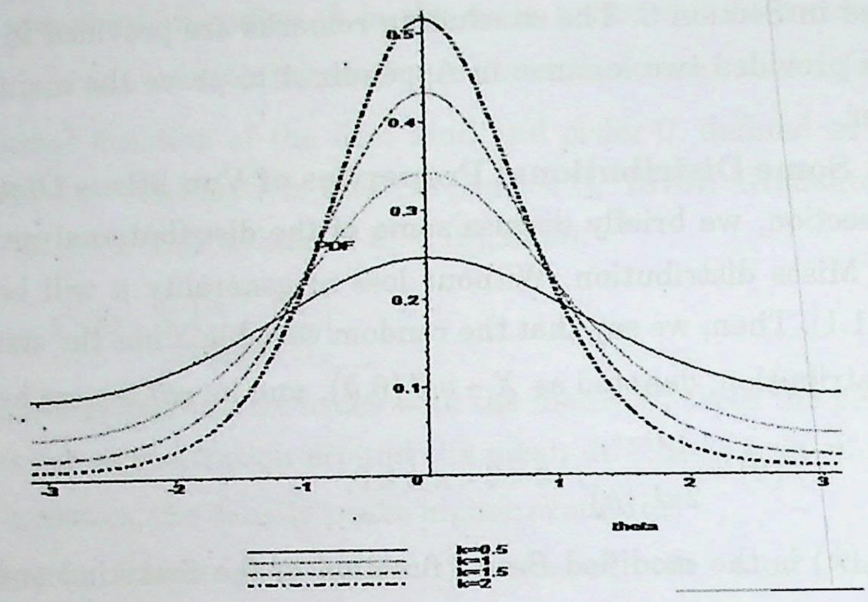

Figure 2.1 : Plots of the Pdf of the standard von Mises distribution,
$X \sim vM(\theta, k)$

In what follows, we will consider several distributional properties of $X \sim vM(\theta, k)$. Based on these distributional properties, some characterizations of $X \sim vM(\theta, k)$ will be given.

The standard von Mises distribution, $X \sim vM(\theta, k)$ with the *pdf* as given in (2.1), has the following properties.

I. It is symmetric around $\theta = 0$.

II. For $k=0$ in (2.1) X has the uniform distribution on $(-\pi, \pi)$.

III. If $k \to \infty$ in (2.1), then X has the normal distribution on $(-\infty, \infty)$, with the *pdf* given by $f(\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\theta^2/2\sigma^2}$, where $\sigma^2 = \frac{1}{k}$.

IV. The mode is at $\theta = 0$, and it is $\frac{e^k}{2\pi I_0(k)}$.

V. $k = ln\left(\frac{f(0)}{f(\pi/2)}\right)$.

VI. Hazard Rate Function. Using the equations (2.2) and (2.3), the hazard rate (or failure rate) $h(\theta)$ of the standard von Mises distribution, $X \sim vM(\theta,k)$, is given by

$$h(\theta)=\frac{f(\theta)}{1-F(\theta)}=\frac{I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos j\theta}{(2\pi-\theta)I_0(k)-2\sum_{j=1}^{\infty}\frac{I_j(k)}{j}\sin j\theta}. \tag{2.4}$$

To describe the shapes of the hazard rate (or failure rate) of the standard von Mises distribution, $X \sim vM(\theta,k)$, the plots of the hazard rate (2.4) for some values of parameter k are provided below in Figure 2.2. The effects of the parameter can easily be seen from these graphs. Similar plots can be drawn for other values of the parameters.

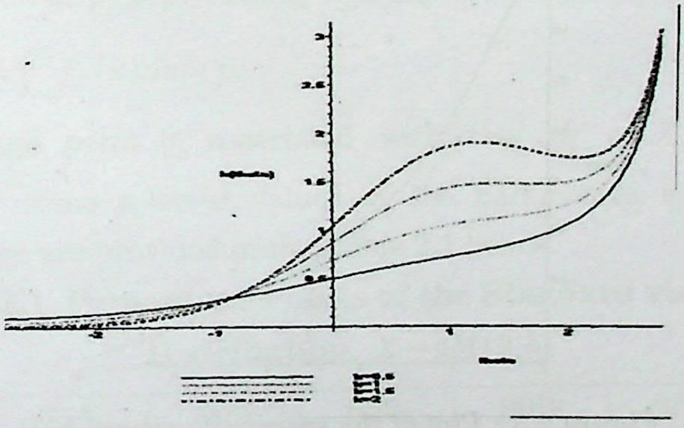

Figure 2.2 : Plots of the hazard rate $h(\theta)$ of the von Mises distribution, $X \sim vM(\theta,k)$

VII. Shannon Entropy. An entropy provides an excellent tool to quantity the amount of information (or uncertainty) contained in random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. As proposed by Shannon ([21], 1948), If X is a continuous random variable with *pdf* $f_X(x)$, defined over an interval Ω then Shannon's entropy of X, denoted by $H(f)$, is defined as

$$H(f)=E\left[-ln\{f_X(x)\}\right]=\int_{\Omega}f_X(x)ln\{f_X(x)\}dx. \quad (2.5)$$

Now, using the *pdf* (2.1) of the standard von Mises distribution, $X \sim vM(\theta,k)$, in Eq. (2.5), and integrating with respect to θ and simplifying, we obtain an explicit expression of Shannon entropy as follows:

$$H(f)=ln\left(2\pi I_0(k)\right)-\frac{kI_1(k)}{I_0(k)}, k\geq 0, \quad (2.6)$$

where $I_0(k)$ and $I_1(k)$ are the modified Bessel function of the first kind of orders 0 and 1 respectively. Obviously, $H(f)$ is a function of k. To describe the shapes of the entropy function, $H(f)$, the plot of the entropy (2.6) for some values of the parameter k are provided below in Figure 2.3.

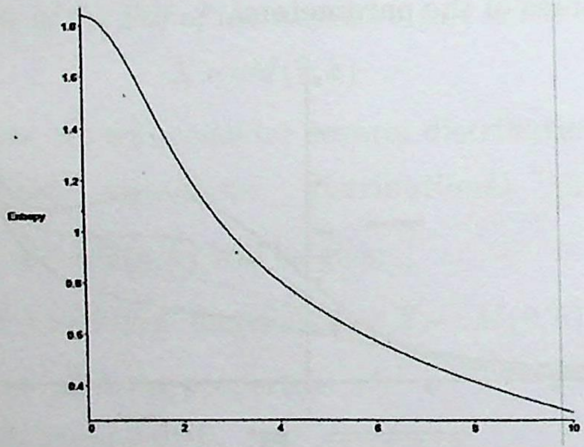

Figure 2.3 : Plot of the entropy function, H(f)

Since $\frac{d}{dk}I_0(k)=I_1(k), \frac{d}{dk}I_1(k)=I_0(k)-\frac{I_1(k)}{k}$, and $I_1(k)<I_0(k), \forall k\geq 0$, (see Oldham, et al. (2002), for details) , we observe that the entropy function, $H(f)$, is a monotonic decreasing function of k, which follows by differentiating Eq. (2.6) with respect to k, and noting that $\frac{d}{dk}H(f)<0, \forall k\geq 0$. Further, by actual computation (using Maple software), it has been found that $\frac{d^2}{dk^2}H(f)=-0.63\times 10^{-8}$, when

k=1.3344523 and $\frac{d^2}{dk^2}H(f)=0.247\times10^{-7}$, when $k=1.3344524$, the entropy function, $H(f)$, has an inflection point, say, $k=\alpha$, in the interval (1.3344523, 1.3344524), that is, $\frac{d^2}{dk^2}H(f)<0$, when $k<\alpha$, and $\frac{d^2}{dk^2}H(f)<0$ when $k>\alpha$, which implies that the entropy function, $H(f)$ is concave down when $k<\alpha$, and concave up when $k>\alpha$.

VIII. Percentile Points. Here we compute the percentage points of the standard von Mises distribution, $X \sim vM(k)$, with the *pdf* as given in (2.1) and *cdf* as given in (2.3). For any 0<p<1, the 100*pth* percentile (also called the quantile of order p) of the standard von Mises distribution, $X \sim vM(\theta,k)$, with *pdf* $f_X(\theta)$ is a number θ_p such that the area under $f_X(\theta)$ to the left of θ_p is p. That is, θ_p is any root of the equation given by

$$F(\theta_p)=\int_{-\infty}^{x_p} f_X(u)\,du=p.$$

The percentage point θ_p associated with the *cdf* of $X \sim vM(\theta,k)$ are computed for some selected values of the parameters by using Maple software. These are provided in the Table 2.1 below.

Table 2.1. Percentage Points of the Standard von Mises Distribution, $X \sim vM(\theta,k)$

Parameter k	75%	80%	85%	90%	95%	99%
0.1	4.812	5.119	5.418	5.710	5.998	6.226
0.2	4.909	5.206	5.488	5.759	6.023	6.231
0.5	5.171	5.424	5.656	5.872	6.080	6.243
1.0	5.473	5.661	5.829	5.986	6.136	6.254
1.5	5.649	5.794	5.829	6.049	6.167	6.260
2.0	5.753	5.874	5.984	6.087	6.186	6.264
2.5	5.821	5.926	6.021	6.111	6.198	6.266
3.0	5.870	5.962	6.048	6.129	6.206	6.268

3.5	5.904	5.990	6.067	6.142	6.213	6.269
4.0	5.931	6.010	6.083	6.152	6.218	6.270
4.5	5.953	6.027	6.095	6.160	6.222	6.271
5.0	5.971	6.041	6.106	6.167	6.225	6.272
5.5	5.987	6.053	6.114	6.172	6.228	6.272
6.0	6.000	6.064	6.122	6.177	6.231	6.273
6.5	6.012	6.073	6.129	6.182	6.233	6.273
7.0	6.022	6.081	6.134	6.185	6.235	6.274
7.5	6.032	6.088	6.140	6.189	6.236	6.274
8.0	6.040	6.094	6.144	6.192	6.238	6.274
8.5	6.048	6.100	6.149	6.195	6.239	6.274
9.0	6.054	6.105	6.153	6.197	6.241	6.275
9.5	6.061	6.110	6.156	6.200	6.242	6.275
10.0	6.067	6.115	6.160	6.202	6.243	6.275

3. Characterizations of the von Mises Distribution. Characterization of probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real-world data, it is necessary to confirm whether the given continuous probability distribution satisfies the underlying requirements by its. For various methods of characterizing a probability distribution, the interested readers are referred to Nagaraja ([18],2006), and Ahsanullah et al. ([2],2014), among others. It appears from the literature that not much attention has been paid to the characterization of the von Mises distribution which can be usefully employed; see Best and Fisher ([6] 1979). For the maximum likelihood characterization of the von Mises distribution , the interested readers are referred to Bingham and Mardia ([7] 1975). The section presents characterizations of the von Mises distribution by truncated first moment, order statistics and upper record values. For this, we will need to following assumption and lemmas as provided below.

Assumption 3.1. Suppose the random variable X is absolutely continuous with cumulative distribution function (*cdf*) $F(x)$ and probability density

function (*pdf*) $f(x)$. We assume that $\gamma = \{x \mid F(x) > 0\}$ and $\delta = \inf\{x \mid F(x) < 1\}$. We further assume that $E(X)$ exists.

Lemma 3.1. Under the assumption 3.1, if $E(X \mid X \le x) = g(x)\tau(x)$, where $\tau(x) = f(x)/F(x)$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_\gamma^x \frac{u - g'(u)}{g(u)} du$ is finite for all $x, \gamma < x < \delta$, then $f(x) = ce^{\int_\gamma^x \frac{u - g'(u)}{g(u)} du}$, where c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.

Proof. We have $g(x) = \frac{\int_\gamma^x f(u)du}{f(x)}$, that is, $\int_\gamma^x uf(u)du = f(x)g(x)$.

Differentiating the above equation with respect to respect to x, we obtain

$$xf(x) = f'(x)g(x) + f(x)g'(x).$$

From the above equation, we obtain

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}.$$

On integrating the above equation with respect to x, we have

$$f(x) = ce^{\int_\gamma^x \frac{u - g'(u)}{g(u)} du},$$

where c is obtained by the condition $\int_\gamma^\delta f(x)dx = 1$. This completes the proof of Lemma 3.1.

Lemma 3.2. Under the assumption 3.1, if $E(X \mid X \ge x) = g(x)r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_x^\delta \frac{u + g'(u)}{g(u)} du$ is finite for all $x, \gamma < x < \delta$, then $f(x) = ce^{-\int_\gamma^x \frac{u + g'(u)}{g(u)} du}$, where c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.

Proof. We have $g(x)=\dfrac{\int_x^\delta uf(u)du}{f(x)}$, that is, $\int_x^\delta uf(u)du = f(x)g(x)$.

Differentiating the above equation with respect to respect to x, we obtain

$$-xf(x)=f'(x)g(x)+f(x)g'(x).$$

From the above equation, we obtain

$$\frac{f'(x)}{f(x)}=-\frac{x+g'(x)}{g(x)}.$$

On integrating the above equation with respect to x, we have

$$f(x)=ce^{-\int_\gamma^x \frac{u+g'(u)}{g(u)}du},$$

where c is obtained by the condition $\int_\gamma^\delta f(x)dx=1$. This completes the proof of Lemma 3.2.

3.1. Characterization by Truncated First Moment. This section presents characterizations of the von Mises distribution by truncated first moment, as provided in Theorem 3.1.1 and 3.1.2 below. Without loss of generality, we will consider the following *pdf* $f(x)$ of the standard von Mises distribution, $X \sim vM(\theta,k)$, expressed as a series of modified Bessel functions:

$$f(\theta)=\frac{e^{k\cos\theta}}{2\pi I_0(k)}=\frac{1}{2\pi}\left(1+\frac{2}{I_0(k)}\sum_{j=1}^{\infty}I_j(k)\cos j\theta\right),-\pi<\theta<\pi,k\geq 0,$$

as provided in Equations (2.1) and (2.2) above.

Theorem 3.1.1. Suppose that X is absolutely continuous bounded random variable with *cdf* $F(x)$ such that $F(-\pi)=0$ and $F(\pi)=1$ then $E(X\mid X<\theta)=g(\theta)\tau(\theta)$, where $\tau(\theta)=\dfrac{f(\theta)}{F(\theta)}$ and $g(\theta)$ is a continuous differentiable function of $\theta, -\pi<\theta<\pi$, given by

$$g(\theta)=e^{-k\cos\theta}\left(\frac{\theta^2-\pi^2}{2}I_0(k)+2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(j\theta\sin j\theta+\cos j\theta-\cos j\pi)\right),$$

if and only if

$$f(\theta)=\frac{1}{2\pi I_0(k)}e^{k\cos\theta},-\pi<\theta<\pi,k\geq 0.$$

Proof. If $f(\theta)=\frac{1}{2\pi I_0(k)}e^{k\cos\theta},-\pi<\theta<\pi,k\geq 0,$ then, expressing it as a series of modified Bessel functions, it is easily seen, after integration and simplification, that

$$g(\theta)=\frac{\int_{-x}^{\theta}uf(u)du}{f(\theta)}=\frac{\int_{-x}^{\theta}u\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos ju\right)du}{e^{k\cos\theta}}$$

$$=e^{-k\cos\theta}\left[\left\{\left(\theta^2-\pi^2\right)I_0(k)+2\sum_{j=1}^{\infty}\frac{I_j(k)}{j}\theta\sin j\theta\right\}\right.$$

$$\left.-\left\{\frac{\left(\theta^2-\pi^2\right)}{2}I_0(k)-2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(\cos j\theta-\cos j\pi)\right\}\right]$$

$$=e^{-k\cos\theta}\left(\frac{\theta^2-\pi^2}{2}I_0(k)+2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(j\theta\sin j\theta+\cos j\theta-\cos j\pi)\right).$$

Suppose that

$$g(\theta)=e^{-k\cos\theta}\left(\frac{\theta^2-\pi^2}{2}I_0(k)+2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(j\theta\sin j\theta+\cos j\theta-\cos j\pi)\right).$$

Then, differentiating both sides of the above equations with respect to θ and simplifying, it is easily seen that

$$g'(\theta)=\theta+k\sin\theta g(\theta),\ \text{since}\ e^{k\cos\theta}=\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos j\theta\right).$$

Thus

$$\frac{\theta-g'(\theta)}{g(\theta)}=-k\sin\theta,$$

from which, on using Lemma 3.1, we have

$$\frac{f'(\theta)}{f(\theta)}=\frac{\theta-g'(\theta)}{g(\theta)}=-k\sin\theta.$$

On integrating the above equation with respect to θ, we obtain

$f(\theta) = ce^{k\cos\theta}$, where c is a constant to be determined.

Using the condition $\int_{-\pi}^{\pi} f(u)du = 1$, and recalling the integral representation of the modified Bessel function of the first kind and order 0, we easily obtain $c = \frac{1}{2\pi I_0(k)}$, and thus

$$f(\theta) = \frac{1}{2\pi I_0(k)} e^{k\cos\theta}, -\pi < \theta < \pi, k \geq 0,$$

which is the *pdf* of the standard von Mises distribution $X \sim vM(\theta, k)$. The completes the proof of the Theorem 3.1.1.

Theorem 3.1.2. Suppose that X is absolutely continuous bounded random variable with *cdf* $F(x)$ such that $F(-\pi) = 0$ and $F(\pi) = 1$, then $E(X \mid X > \theta) = h(\theta) r(\theta)$, where $r(\theta) = \frac{f(\theta)}{1 - F(\theta)}$ and $h(\theta)$ is a continuous differentiable function of $\theta, -\pi < \theta < \pi$, given by

$$h(\theta) = e^{-k\cos\theta}\left(\frac{x^2 - \theta^2}{2} I_0(k) - 2\sum_{j=1}^{\infty} \frac{I_j(k)}{j^2}(j\theta \sin j\theta + \cos j\theta - \cos j\pi)\right),$$

if and only if

$$f(\theta) = \frac{1}{2\pi I_0(k)} e^{k\cos\theta}, -\pi < \theta < \pi, k \geq 0.$$

Proof. Suppose that

$$f(\theta) = \frac{1}{2\pi I_0(k)} e^{k\cos\theta} = \frac{1}{2\pi I_0(k)}\left(I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos j\theta\right), -\pi < \theta < \pi, k \geq 0,$$

then

$$h(\theta) = \frac{\int_\theta^x f(u)du}{f(\theta)} = \frac{\int_0^x u\left(I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos ju\right)du}{e^{k\cos\theta}}$$

$$= e^{-k\cos\theta}\left[\left\{\left(\pi^2 - \theta^2\right) I_0(k) - 2\sum_{j=1}^{\infty} \frac{I_j(k)}{j}\theta \sin j\theta\right\}\right.$$

$$-\left\{\frac{\left(\pi^2-\theta^2\right)}{2}I_0(k)-2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(\cos j\pi-\cos j\theta)\right\}\Bigg]$$

$$=e^{-k\cos\theta}\left(\frac{\pi^2-\theta^2}{2}I_0(k)-2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(j\theta\sin j\theta+\cos j\theta-\cos j\pi)\right).$$

Suppose that

$$h(\theta)=e^{-k\cos\theta}\left(\frac{\pi^2-\theta^2}{2}I_0(k)-2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}(j\theta\sin j\theta+\cos j\theta-\cos j\pi)\right).$$

Then, differentiating both sides of the above equations with respect to θ and simplifying, it is easily seen that

$$h'(\theta)=k\sin\theta h(\theta)-\theta,\ \text{since}\ e^{k\cos\theta}=\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos j\theta\right).$$

Thus

$$\frac{\theta+h'(\theta)}{h(\theta)}=k\sin\theta,$$

from which, on using Lemma 3.2, we have

$$\frac{f'(\theta)}{f(\theta)}=-\frac{\theta+h'(\theta)}{h(\theta)}=-k\sin\theta.$$

On integrating the above equation with respect to θ, we obtain $f(\theta)=ce^{k\cos\theta}$, where c is a constant to be determined.

Thus, taking $\gamma=-\pi$ and $\delta=\pi$ in Lemma 3.2, and using the condition $\int_{-\pi}^{\pi}f(u)du=1$, and recalling the integral representation of the modified Bessel function of the first kind and order 0, we easily obtain $c=\frac{1}{2\pi I_0(k)}$, and thus

$$f(\theta)=\frac{1}{2\pi I_0(k)}e^{k\cos\theta},-\pi<\theta<\pi,k\geq 0,$$

which is the *pdf* of the standard von Mises distribution, $X\sim vM(\theta,k)$. This completes the proof of the Theorem 3.1.2.

3.2 Characterization by Order Statistics. This section presents charactrizations of the von Mises distribution by order statistics, as provided in Theorems 3.2.1 and 3.2.2 below. Without loss of generality, we will consider the following *pdf* $f(x)$ of the standard von Mises distribution, $X \sim vM(\theta, k)$:

$$f(\theta)=\frac{1}{2\pi I_0(k)}e^{k\cos\theta}=\frac{1}{2\pi I_0(k)}\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos j\theta\right),-\pi<\theta<\pi,k\geq 0,$$

as provided in Equation (2.1) and (2.2) above.

Let $X_1, X_2, \ldots, X_n$ be n independent copies of the random variable X having absolutely continuous distribution function F(x) and pdf $f(x)$.

Suppose that $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ are the corresponding order statistics. It is known that $X_{j,n} \mid X_{k,n} = x$, for $1 \leq k < j \leq n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v \mid x)$ where $f_V(v \mid x)=\dfrac{f(v)}{1-F(x)}, 0 \leq v < x$, see, for example, Ahsanullah et al. ([2] 2013), chapter 5, or Arnold et al. ([4], 2005), chapter 2, among others. Further, $X_{i,n} \mid X_{k,n} = x, 1 \leq i < k \leq n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w \mid x)$, where $f_W(w \mid x)=\dfrac{f(w)}{F(x)}, w < x$.

We assume that $S_{k-1}=\dfrac{1}{k-1}\left(X_{1,n}+X_{2,n}+\ldots+X_{k-1,n}\right)$, and $T_{k,n}=\dfrac{1}{n-k}\left(X_{k+1,n}+X_{k+2,n}+\ldots+X_{n,n}\right)$, for $1 \leq k < n$.

The following theorem gives a characterization of $X \sim vM(\theta, k)$ based on $S_{k-1}=\dfrac{1}{k-1}\left(X_{1,n}+X_{2,n}+\ldots+X_{k-1,n}\right), 1 \leq k < n.$

Theorem 3.2.1. Suppose that X is absolutely continuous bounded random variable with *cdf* $F(x)$ such that $F(-\pi)=0$ and $F(\pi)=1$, and $S_{k-1}=\dfrac{1}{k-1}\left(X_{1,n}+X_{2,n}+\ldots+X_{k-1,n}\right), 1\le k<n$, then

$$E\left(S_{k-1}\mid X_{k,n}=\theta\right)=e^{-k\cos\theta}\left(\frac{\theta^2-\pi^2}{2}I_0(k)+2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}\left(j\theta\sin j\theta+\cos j\theta-\cos j\pi\right)\right),$$

if and only if

$$f(\theta)=\frac{1}{2\pi I_0(k)}e^{k\cos\theta}, -\pi<\theta<\pi, k\ge 0.$$

Proof. Since $E\left(S_{k-1}\mid X_{k,n}=\theta\right)=\dfrac{\int_{-x}^{\theta}uf(u)du}{f(\theta)}=\dfrac{\int_{-x}^{\theta}u\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos ju\right)du}{e^{k\cos\theta}}$,

the proof easily follows from the Theorem 3.1.1.

The following theorem gives a charactrization of $X\sim vM(\theta,k)$ based on

$$T_{k,n}=\frac{1}{n-k}\left(X_{k+1,n}+X_{k+2,n}+\ldots+X_{n,n}\right), 1\le k<n.$$

Theorem 3.2.2. Suppose that X is absolutely continuous bounded random variable with cdf$F(x)$ such that $F(-\pi)=0$ and $F(\pi)=1$, and $T_{k,n}=\dfrac{1}{n-k}\left(X_{k+1,n}+X_{k+2,n}+\ldots+X_{n,n}\right)$, $1\le k<n$. Then

$$E\left(T_{k,n}\mid X_{k,n}\right)=e^{-k\cos\theta}\left(\frac{\pi^2-\theta^2}{2}I_0(k)-2\sum_{j=1}^{\infty}\frac{I_j(k)}{j^2}\left(j\theta\sin j\theta+\cos j\theta-\cos j\pi\right)\right),$$

if and only if

$$f(\theta)=\frac{1}{2\pi I_0(k)}e^{k\cos\theta}, -\pi<\theta<\pi, k\ge 0.$$

Proof. Since $E\left(T_{k,n}\mid X_{k,n}\right)=\dfrac{\int_{\theta}^{x}uf(u)du}{f(\theta)}=\dfrac{\int_{\theta}^{\pi}u\left(I_0(k)+2\sum_{j=1}^{\infty}I_j(k)\cos ju\right)du}{e^{k\cos\theta}}$,

the proof easily follows from the Theorem 3.1.2.

3.3. Characterization by Upper Record Values. This section presents characterizations of the von Mises distribution by upper record values, as provided in Theorem 3.3.1 below. Without loss of generality, we will consider the following pdf $f(x)$ of the standard von Mises distribution, $X \sim vM(\theta, k)$:

$$f(\theta) = \frac{1}{2\pi I_0(k)} e^{k\cos\theta} = \frac{1}{2\pi I_0(k)}\left(I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos j\theta\right), -\pi < \theta < \pi, k \geq 0,$$

as provided in Equation (2.1) and (2.2) above.

Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. Let $Y_n = \max(X_1, X_2, \ldots, X_n)$ for $n \geq 1$. We say that X_j is an upper record value of $\{X_n, n \geq 1\}$ if $Y_j > Y_{j-1}, j > 1$. The indices as which the upper records occur are given by the record times $\{U(n) > \min(j \mid j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. We will denote the nth upper record value as $X(n) = X_{U(n)}$.

Theorem 3.3.1. Suppose the random variable X satisfies the assumption 3.1 with $\gamma = -\pi$ and $\delta = \pi$. Then X has the standard von Mises distribution, $X \sim vM(\theta, k)$, if and only if

$$E(X(n+1) \mid X(n) = \theta)$$

$$= e^{-k\cos\theta}\left(\frac{\pi^2 - \theta^2}{2} I_0(k) - 2\sum_{j=1}^{\infty} \frac{I_j(k)}{j^2}(j\theta \sin j\theta + \cos j\theta - \cos j\pi)\right).$$

Proof. Since

$$E(X(n+1) \mid X(n) = \theta)$$

$$= \frac{\int_{\theta}^{x} uf(u)\,du}{1 - F(\theta)}$$

$$= E(X \mid X \geq \theta)$$

$$= \frac{\int_{\theta}^{x} u\left(I_0(k) + 2\sum_{j=1}^{\infty} I_j(k)\cos ju\right)du}{e^{k\cos\theta}},$$

the proof easily follows from the Theorem 3.1.2.

4. Concluding Remarks. Characterization of a probability distribution plays an important role in probability and statistics, and other applied sciences. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various methods. The von Mises distribution is one of the most important distributions in statistics to deal with the circular or directional data. In this paper, motivated by the importance of the von Mises distribution in many practical problems in circular or directional data, we will consider its several distributional properties. Based on these distributional properties, we have established some new characterizations of the von Mises distribution by truncated first moment, order statistics and upper record values. It is hoped that the findings of the paper will be useful for researchers in the fields or probability; statistics, and other applied sciences.

ACKNOWLEDGEMENTS

Part of the paper was completed while the first author was Visiting Professor M. Ahsanullah, Rider University, Lawrenceville, NJ 08648, USA, during the Summer of 2013, and aslo when he took independent studies and research with Professor M. Ahsanullah at Rider University. He is grateful to Miami Dade College for all the support, including tuition grants.

Declaration. We confirm that none of the authors have any competing interests in the manuscript.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

ON THE COEFFICIENTS OF α-LOGARITHMICALLY CONVEX FUNCTIONS

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(*Recieved : October* 20, 2015)

ABSTRACT

For $\alpha \geq 0$, let M^{α} be the set of α-logarithmically convex functions i.e. f analytic in $z \in D = \{z : |z| < 1\}$ satisfying

$$\mathrm{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\right] > 0.$$

Some sharp bounds for the initial coefficients of the inverse function f^{-1} of $f \in M^{\alpha}$ are given.

2000 AMS Subject Classification: Primary 30C45; Secondary 30C50
Keywords: Ünivalent functions, starlike, convex, α-logarithmically convex functions, inverse functions, coefficients.

1. Introduction. Let S be the class of analytic normalised univalent functions f, defined in $z \in D = \{z : |z| < 1\}$ and given by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

Denote by S^* the subset of functions, starlike with respect to the origin and by C the subset of convex functions.

Thus $f \in S^*$ if, and only if, for $z \in D$,

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0,$$

and $f \in C$ if, and only if,

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0.$$

For $\alpha \geq 0$, let M^{α} denote the class of functions satisfying

$$\text{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\right]>0$$

for $z \in D$.

We note that the class M^{α} is the power analogue of the so-called Ma-Minda functions M^{α}, which have been widely studied [8]. It was shown in [1] that for $\alpha \geq 0, M^{\alpha} \subset S^*$ and, together with other results, sharp upper bounds for $|a_2|, |a_3|$ and the Fekete-Szegö functional where found.

Since functions in M^{α} are univalent in D, they possess an inverse function f^{-1}, defined on some set of $f(D)$ with radius at least 1/4.

Suppose that on such a set, f^{-1} has Taylor series expansion

$$f^{-1}(\omega)=\omega+\sum_{n=2}^{\infty}\gamma_n\omega^n. \tag{1.2}$$

It is the main purpose of this paper to find some sharp upper bounds of $|\gamma_n|$ for $n = 2$, 3 and 4 for $f \in M^{\alpha}$.

Let $h \in P$, the class of functions with positive real part in D and write

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n.$$

We shall make use the following well-known results [2,6]:

Lemma 1. If $p \in P$, then $|c_n| \leq 2$ for $n \geq 1$, and

$$\left|c_2-\frac{\mu}{2}c_1^2\right| \leq \max\{2, 2|\mu-1|\} = \begin{cases} 2 & 0 \leq \mu \leq 2, \\ 2|\mu-1| & \text{elsewhere.} \end{cases}$$

Lemma 2. If $h \in P$ with coefficients c_n as above, then for some complex valued x with $|x| \leq 1$ and some complex valued ζ with $|\zeta| \leq 1$

$$2c_2 = c_1^2 + x(4-c_1^2)$$

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)\zeta$$

We prove the following:

2. Theorem. Let $f \in M^{\alpha}$ and f^{-1} be given by (1.2). Then the following sharp inequalities hold

$$|\gamma_2| \leq \frac{2}{1+\alpha},$$

$$|\gamma_3| \le \frac{5+7\alpha}{(1+a)^2(1+2a)},$$

$$|\gamma_3| \le \frac{5+7\alpha}{(1+a)^2(1+2a)},$$

$$|\gamma_4| \le \frac{2(63+77a+3a^2+a^3)}{9(1+a)^3(1+3a)}.$$

The inequalities for $|\gamma_2|$ and $|\gamma_3|$ are valid when $\alpha \ge 0$ and for $|\gamma_4|$ when $0 \le \alpha <\approx 2.04$.

Proof. Write

$$(2.1) \quad \left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = h(z).$$

Since $f\left(f^{-1}(\omega)\right) = \omega$, then (1.1) and (1.2) give

$$(2.2) \quad \begin{aligned} \gamma_2 &= -a_2, \\ \gamma_3 &= -a_3 + 2a_2^2, \\ \gamma_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \end{aligned}$$

and equating coefficients in (2.1), we obtain

$$(2.3) \quad \begin{aligned} a_2 &= \frac{c_1}{1+\alpha}, \\ a_3 &= \frac{(2+7\alpha-\alpha^2)c_1^2}{4(1+\alpha)^2(1+2\alpha)} + \frac{c_2}{2(1+2\alpha)}, \\ a_4 &= \frac{(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)c_1^3}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)} + \frac{(3+19\alpha-4\alpha^2)c_1c_2}{6(1+\alpha)(1+2\alpha)} + \frac{c_3}{3(1+3\alpha)} \end{aligned}$$

Since $|c_1| \le 2$, it follows at once from (2.2) and (2.3) that $|\gamma_2| \le 2/(1+\alpha)$.

For γ_3, (2.2) and (2.3) give

$$(2.4) \quad |\gamma_3| = \left|\frac{c_2}{2(1+2\alpha)} - \frac{(6+9\alpha+\alpha^2)c_1^2}{4(1+\alpha)^2(1+2\alpha)}\right|.$$

Applying Lemma 1 with $\mu = \dfrac{6+9\alpha+\alpha^2}{2(1+\alpha)^2}$, so that $\mu \notin [0,2]$, we obtain

$$|\gamma_3| \le \frac{(5+7\alpha)}{(1+\alpha)^2(1+2\alpha)},$$

which proves the second inequality in the Theorem.

For γ_4, we obtain from (2.2) and (2.3)

$$(2.5)\quad |\gamma_4| = \left| \frac{c_3}{3(1+3\alpha)} - \frac{(6+\alpha)c_1c_2}{3(1+\alpha)(1+3\alpha)} + \frac{(48+73\alpha+21\alpha^2+2\alpha^3)c_1^3}{18(1+\alpha)^3(1+3\alpha)} \right|.$$

We now use Lemma 2 to express c_3 and c_2 in terms of c_1 to obtain

$$|\gamma_4| = \left| \frac{(63+77\alpha+3\alpha^2+\alpha^3)c_1^3}{36(1+\alpha)^3(1+3\alpha)} - \frac{5c_1x(4-c_1^2)}{6(1+\alpha)(1+3\alpha)} - \frac{c_1x^2(4-c_1^2)}{12(1+3\alpha)} + \frac{(4-c_1^2)(1-|x|^2)\zeta}{6(1+3\alpha)} \right|.$$

Without loss in generality we may normalise the coefficient c_1 and assume that $c_1 = c$, where $0 \le c \le 2$. Then using the triangle inequality we obtain

$$|\gamma_4| \le \frac{(63+77\alpha+3\alpha^2+\alpha^3)c^3}{36(1+\alpha)^3(1+3\alpha)} - \frac{5c|x|(4-c^2)}{6(1+\alpha)(1+3\alpha)} + \frac{c|x|^2(4-c^2)}{12(1+3\alpha)} + \frac{(4-c^2)(1-|x|^2)}{6(1+3\alpha)} = \psi(c,|x|).$$

Assume now that $\psi(c,|x|)$ has a critical point inside $[0,\ 2]\times[0,\ 1]$, then differentiating with respect to $|x|$ and equating to zero implies $c = 2$, which is a contradiction.

Thus in order to find the maximum of $\psi(c,|x|)$, we need only consider the end points of $[0,\ 2] \times [0,\ 1]$.

On $c = 0$,

$$\psi(0,|x|) = \frac{2(1-|x|^2)}{3(1+3\alpha)} \le \frac{2}{3(1+3\alpha)}.$$

On $c = 2$,

$$\psi(2,|x|) = \frac{2(63+77\alpha+3\alpha^2+\alpha^3)}{9(1+\alpha)^3(1+3\alpha)}.$$

On $|x| = 0$,

$$\psi(c,0) = \frac{(63+77\alpha+3\alpha+\alpha^3)c^3}{36(1+\alpha)^3(1+3\alpha)} + \frac{(4-c^2)}{6(1+3\alpha)}.$$

This expression is minimum at $p=0$, and maximum at

$$p = \frac{4(1+\alpha)^3}{(63+77\alpha+3\alpha^2+\alpha^3)} \text{ for } \alpha \ge 0.$$

Substituting back gives

$$\psi(c,0) \le \frac{2(63+77\alpha+3\alpha+\alpha^3)}{9(1+\alpha)^3(1+3\alpha)}.$$

Finally on $|x|=1$,

$$\psi(c,1)\le\frac{(63+77\alpha+3\alpha^2+\alpha^3)c^3}{36(1+\alpha)^3(1+3\alpha)}+\frac{(4-c^2)c}{12(1+3\alpha)}+\frac{5(4-c^2)}{6(1+\alpha)(1+3\alpha)}.$$

This expression increase with 2 on [0, 2] provided $0\le\alpha\approx 2.04$, which again gives the inequality for $|\gamma_4|$.

Taking $c_1=c_2=c_3=2$ in (2.4) and (2.5) shows that the inequalities in the Theorem are sharp.

Remark 1. The analysis shows that the inequalities for $|\gamma_2|$ and $|\gamma_3|$ are valid for $\alpha\ge 0$, and for $|\gamma_4|$ when $0\le\alpha\approx 2.04$, Finding the sharp inequality for $|\gamma_4|$ for all $\alpha\ge 0$ remains an open question.

Remark 2. When $\alpha=0$, the inequalities obtained in the Theorem are consistent with those given by Lowner [7] for $f\in S$, and when $\alpha=1$, with those given in [5] for convex functions.

Remark 3. For real μ and ν, the functional $J_4(f)=\left|a_4+\mu a_2a_3+\upsilon a_2^3\right|$ plays an important role in the theory of univalent functions. In [9], sharp bounds for $J_4(f)$ for all real μ and ν were obtained for the Ma-Minda functions M_α. This paper finds the sharp bound for $J_4(f)$ for functions in M^α in the case $\mu=-5$ and $\nu=5$, (for the fourth inverse coefficient). Finding sharp bounds for $J_4(f)$ for all real μ and ν when $f\in M^\alpha$ remains an open problem. In particular obtaining the exact upper bound for $J_4(f)$ when $\mu=-1$ and $\nu=1/3$ would provide a solution to the problem of finding the sharp bound for the third coefficient of $\log(f(z)/z)$ for $f\in M^\alpha$. However the method used in this paper does not appear to give the sharp bound in this case.

Remark 4. When $0\le\alpha\le 1$, using the inequalities $|c_n|\le 2$ for $n=1,2,3$ in (2.3) above and a similar expression for a_5 in terms of c_1,c_2,c_3 and c_4, easily establishes the following, which are sharp when $c_1=c_2=c_3=c_4=2$.

$$|a_2|\le\frac{2}{1+\alpha},$$

$$|a_3|\le\frac{3(1+3\alpha)}{(1+\alpha)^2(1+2\alpha)},$$

$$|a_4|\le\frac{2(18+113\alpha+292\alpha^2+7\alpha^3+2\alpha^4)}{9(1+\alpha)^3(1+2\alpha)(1+3\alpha)},$$

$$|a_5|\le\frac{45+629\alpha+2908\alpha^2+2969\alpha^3+12061\alpha^4+812\alpha^5+196\alpha^6}{9(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}.$$

Following the same procedure as above gives sharp upper bounds for $|a_n|$ for $n \geq 6$ for functions in M^{α} which are computable, but the calculations become increasingly complicated.

Kulshrestha [3, 4], found sharp upper bounds for the coefficients of functions in M_{α} for $n \geq 2$, but since functions in M^{α} involve powers, solving the corresponding problem for M^{α} for all $n > 2$ may be more difficult.

We finally note that finding the correct order of growth for the coefficients a_n as $n \to \infty$ for $f \in M^{\alpha}$ is also an open problem.

ACKNOWLEDGEMENT

The authors would like to acknowledge and appreciate the financial support received from Universiti Kebangsaan Malaysia(UKM) under the grant: AP-2013-009.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

SOME PROPERTIES OF HARMONIC MULTIVALENT FUNCTIONS ASSOCIATED WITH THE GENERALIZED SALAGEAN OPERATOR

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(Received : December 4, 2015)

ABSTRACT

In this paper, some investigations of a class of the generalized Salagean-type harmonic multivalent functions is presented. Coefficient inequalities, extreme points, distortion theorem, properties of neighborhood and inclusion relationships are investigated in here. Also, we obtain some results on partial sums of an integral operator for multivalent harmonic functions.

2010 Mathematics Subject Classification : 30C45.
Keywords and Phrases : Harmonic functions, multivalent functions, Salagean operator.

1. Introduction. A complex-valued function $f = u + iv$ in a complex domain $D \subset \mathbb{C}$ is said to be harmonic if both u and v are real harmonic in D. If D is a simply connected domain, then $f = h + \overline{g}$, where h and g are analytic in D. The functions h and g are called the analytic part and co-analytic part of f, respectively. A harmonic mapping $f = h + \overline{g}$ is sense preserving in D if and only if the analytic function $w = g'/h'$ satisfies $|w(z)| < 1$ in D.

In [1], Ahuja and Jahangiri defined the class $\mathcal{H}_p\,(p \in \mathbb{N})$ consisting of all multivalent harmonic functions $f = h + \overline{g}$ that are sense-preserving in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and h, g are of the form:

$$h(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \text{ and } g(z) = \sum_{k=0}^{\infty} b_{k+p} z^{k+p}, \tag{1}$$

where $|b_p|<1$. Note that the class $\mathcal{H}_p$ reduces the class $\mathcal{S}_H$ which was introduced by Clunie and Sheil-Small [4]. Also, note that the class $\mathcal{H}_p$ reduces to the class $\mathcal{A}_p$ of analytic multivalent functions, whenever the co-analytic part of harmonic function is identically zero. Furthermore, let the class $\mathcal{NH}_p$ consist of harmonic functions $f=h+\overline{g}\in\mathcal{H}_p$ such that h and g are of the form:

$$h(z)=z^p-\sum_{k=1}^{\infty}|a_{k+p}|z^{k+p} \text{ and } g(z)=-\sum_{k=1}^{\infty}|b_{k+p}|z^{k+p}. \tag{2}$$

Let $p,n\in\mathbb{N}$ and $\lambda\geq 0$. For an analytic function $f(z)\in\mathcal{A}_p$, we define

$D^0_{\lambda,p}f(z)=f(z)$,

$D^1_{\lambda,p}f(z)=(1-\lambda)f(z)+\dfrac{\lambda}{p}zf'(z)=D_{\lambda,p}f(z)$,

$D^2_{\lambda,p}f(z)=D_{\lambda,p}\left(D_{\lambda,p}f(z)\right)$

and

$D^n_{\lambda,p}f(z)=D_{\lambda,p}\left(D^{n-1}_{\lambda,p}f(z)\right)$.

For $p=1$, we get Al-Oboudi differential operator [2], and for $p=\lambda=1$, we get the Salagean differential operator [15].

Several authors defined many subclasses of $\mathcal{H}_p$ using the operator $D^n_{\lambda,p}$ (cf. [11,19]). For the subclasses of univalent harmonic functions using $D^n_{\lambda,p}$, we may refer to [5,9,13,17,18].

Let $p,n\in\mathbb{N},\lambda\geq 0$ and $0\leq\mu\leq p$. For $f(z)\in\mathcal{A}_p$, we define an operator $\Omega^n_{\lambda,p,u}:A_p\to A_p$ by

$$\Omega^n_{\lambda,p,u}(f;z)=\frac{\mu}{p}\left(D^{n+1}_{\lambda,p}f(z)\right)+\left(1-\frac{\mu}{p}\right)\left(D^n_{\lambda,p}f(z)\right).$$

We note that the operator $\Omega^n_{\lambda,p,\mu}$ induces $D^n_{\lambda,p}$ and $D^{n+1}_{\lambda,p}$ for the cases $\mu=0$ and $\mu=p$, respectively. Now, using the operator $\Omega^n_{\lambda,p,\mu}$, we define classes $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$ and $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ which are subclasses of $\mathcal{H}_p$ as follows:

Definition 1. Let $p,n\in\mathbb{N},\lambda\geq 0, 0\leq\mu\leq p$ and $0\leq\alpha<1$. We say that a function $f=h+\overline{g}\in\mathcal{H}_p$ is in the class $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$, if f satisfies the following inequality:

$$\operatorname{Re}\left\{\frac{\Omega^{n}_{\lambda,p,\mu}(h;z)}{z^{p}}+\overline{\left(\frac{\Omega^{n}_{\lambda,p,\mu}(g;z)}{z^{p}}\right)}\right\}>\alpha\left(z\in\mathbb{D}\right).$$

Also, let $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ denote the class of functions $f=h+\overline{g}$ in $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$ so that h and g are given by (2). Namely, $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\equiv\mathcal{HS}_p(n,\lambda,\mu;\alpha)\cap\mathcal{NH}_p$.

The class $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$ gives us the following well-known classes.

Remark 1

(a) For $g\equiv 0,\mu=0$ and $n=0$, we have the subclass of close-to-star functions of order α (See [3]).

(b) For $g\equiv 0,\mu=0,n=0,p=1$ and $\alpha=0$, we have the subclass of close-to-star functions which was introduced by Reade [14].

(c) For $g\equiv 0$ and $\mu=p=\lambda=1$, we have the subclass of functions with bounded turning of order α (See [8]).

(d) For $g\equiv 0,\mu=p=\lambda=1$ and $\alpha=0$, we have the subclass of functions with bounded turning (See [12, 16]).

For $f(z)\in\mathcal{A}_p$ with $p\in\mathbb{N}$, we define an operator $L_p:\mathcal{A}_p\to\mathcal{A}_p$ by

$$L_p(f;z)=\frac{p+1}{z}\int_0^z f(\zeta)\,d\zeta=z^p+\sum_{k=1}^{\infty}\frac{p+1}{k+p+1}a_{k+p}z^{k+p}.$$

Then, we have $L_1\equiv L$, where L is the well-known Libera integral operator. Analogous to define the integral operator L_p, we define an operator $\mathcal{L}_p:\mathcal{H}_p\to\mathcal{H}_p$ by

$$\mathcal{L}_p(f;z)=\frac{p+1}{z}\int_0^z h(\zeta)\,d\zeta+\overline{\left(\frac{p+1}{z}\int_0^z g(\zeta)\,d\zeta\right)},\tag{3}$$

where $f=h+\overline{g}\in\mathcal{H}_p$.

For a given harmonic function $f=h+\overline{g}\in\mathcal{H}_p$ of the form (1), the j-th partial sum f_j of f is given by

$$f_i(z)=z^p+\sum_{k=1}^{j}a_{k+p}z^{k+p}+\overline{\left(\sum_{k=0}^{j}b_{k+p}z^{k+p}\right)}.$$

The main object of the present paper is to investigate various properties (coeficient inequalities, extreme points, distortion theorem, neighborhood properties and inclusion relationships) of harmonic multivalent functions belonging to the class $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$ and $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$. And we find the condition of β so that the partial sum of $\mathcal{L}_p(f;z)$ belongs

to the class $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$, for given $f \in \mathcal{HS}_p(n,\lambda,\mu;\beta)$. A similar problem for the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ is also considered.

2 Main Results. First of all, we find a sufficient condition for the function $f(z)$ of the form (1) belong to the class $\mathcal{HS}_p(n,\lambda,\mu;\alpha)$.

Theorem 1. Let $p,n \in \mathbb{N}, \lambda \geq 1, 0 \leq \mu \leq p$ and $0 \leq \alpha < 1$. And let the function $f = h + \overline{g}$ be such that h and g are given by (1). If f satisfies

$$\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}| + \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}| \leq 1-\alpha, \tag{4}$$

then the function f is a harmonic p-valent, sense-preserving function in $\mathbb{D}$ and $f \in \mathcal{HS}_p(n,\lambda,\mu;\alpha)$.

Proof. Let $z \in \mathbb{D}$. At first, we shall show that f is sense-preserving in $\mathbb{D}$. Since $\lambda \geq 1$,, we have

$$\begin{aligned}\left|\frac{h'(z)}{z^{p-1}}\right| &> p\left(1-\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)|a_{k+p}|\right)\\ &\geq p\left(1-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}|\right).\end{aligned} \tag{5}$$

Similar calculations show that

$$\begin{aligned}\left|\frac{g'(z)}{z^{p-1}}\right| &\leq p\left(\sum_{k=0}^{\infty}\left(\frac{p+k}{p}\right)|b_{k+p}||z|^k\right)\\ &< p\left(\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}|\right).\end{aligned} \tag{6}$$

It follows from (4), (5) and (6) that

$$\begin{aligned}\left|\frac{h'(z)}{z^{p-1}}\right| &> p\left(1-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}|\right)\\ &\geq p\left(\alpha+\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}|\right)\\ &> p\left(\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}|\right)\\ &\geq \left|\frac{g'(z)}{z^{p-1}}\right|.\end{aligned}$$

Thus, we have $|w| = |g'(z)/h'(z)| < 1$ in $\mathbb{D}$. Therefore, f is sense-preserving in $\mathbb{D}$. Now, we show that $f \in \mathcal{HS}_p(n,\lambda,\mu;\alpha)$. Using the fact that $\operatorname{Re}(w) \ge \alpha$ if and only if $|1-\alpha+w| - |1+\alpha-w| \ge 0$, it is sufficient to show that

$$\left|1-\alpha+\left(\Omega^n_{\lambda,p,\mu}(h;z)+\Omega^n_{\lambda,p,\mu}(g;z)\right)/z^p\right| - \left|1+\alpha-\left(\Omega^n_{\lambda,p,\mu}(h;z)+\Omega^n_{\lambda,p,\mu}(g;z)\right)/z^p\right| \ge 0. \quad (7)$$

And, we have

$$\begin{aligned}
&\left|1-\alpha+\left(\Omega^n_{\lambda,p,\mu}(h;z)+\Omega^n_{\lambda,p,\mu}(g;z)\right)/z^p\right| - \left|1+\alpha-\left(\Omega^n_{\lambda,p,\mu}(h;z)+\Omega^n_{\lambda,p,\mu}(g;z)\right)/z^p\right| \\
&= \left|2-\alpha+\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}| + \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}|\right| \\
&\qquad - \left|\alpha-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}| - \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}|\right| \\
&\ge 2-\alpha-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}||z|^k - \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}||z|^k \\
&\qquad -\alpha-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}||z|^k - \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}||z|^k \\
&\qquad > 2\left(1-\alpha-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}| - \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}|\right).
\end{aligned}$$

Hence, the inequality (7) hold, if

$$\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}| + \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}| \le 1-\alpha,$$

and the proof of Theorem 1 is completed.

In the following theorem, it is proved that the condition (4) is also necessary for function $f = h + \overline{g}$ of the form (2).

Theorem 2. Let $p, n \in \mathbb{N}, \lambda \ge 1, 0 \le \mu \le p$ and $0 \le \alpha < 1$ and let $f = h + \overline{g}$ be given by (2). Then $f \in \mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, if and only if

$$\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |a_{k+p}| + \sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n |b_{k+p}| \le 1-\alpha. \quad (8)$$

Proof. From Theorem 1, we need to prove the necessary part of this theorem. Since $f \in \mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, we have

$$\operatorname{Re}\left\{\frac{\Omega^n_{\lambda,p,\mu}(h;z)}{z^p}+\overline{\left(\frac{\Omega^n_{\lambda,p,\mu}(g;z)}{z^p}\right)}\right\}<\alpha\left(z\in\mathbb{D}\right)$$

And this is equivalent to

$$\operatorname{Re}\left\{1-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n|a_{k+p}|z^{k+p}-\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n|b_{k+p}|z^{k+p}\right\}>\alpha\left(z\in\mathbb{D}\right).$$

Choosing the values of z on the positive real part axis and letting $z\to 1^-$, we have

$$1-\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n|a_{k+p}|-\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n|b_{k+p}|\ge\alpha,$$

which is the required condition.

Next, we determine the extreme points of the closed convex hull of $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, denoted by $\overline{co}\left(\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\right)$.

Theorem 3 Let $p,n\in\mathbb{N},\lambda\ge 1,0\le\mu\le p$ and $0\le\alpha<1$ and let $f=h+\overline{g}$ be given by (2). $f\in\overline{co}\left(\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\right)$ if and only if

$$f(z)=\sum_{k=0}^{\infty}x_kh_k(z)+y_kg_k(z), \tag{9}$$

where

$$h_0(z)=z^p,$$

$$h_k(z)=z^p-\frac{(1-\alpha)p^{n+2}}{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}z^{k+p}\ (k=1,2,\ldots),$$

$$g_k(z)=z^p-\frac{(1-\alpha)p^{n+2}}{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}\overline{z}^{k+p}\ (k=0,1,2,\ldots)$$

and $x_k\ge 0,y_k\ge 0$ with $\sum_{k=0}^{\infty}(x_k+y_k)=1$. In particular, the extreme points of $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For the functions of the form (9), we have

$$f(z)=z^p-\sum_{k=1}^{\infty}\frac{(1-\alpha)p^{n+2}}{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}x_kz^{k+p}-\sum_{k=0}^{\infty}\frac{(1-\alpha)p^{n+2}}{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}y_k\overline{z}^{k+p}.$$

Then

$$\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n\frac{(1-\alpha)p^{n+2}}{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}x_k+\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n\frac{(1-\alpha)p^{n+2}}{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}y_k$$

$$=(1-\alpha)\left\{\sum_{k=1}^{\infty}x_k+\sum_{k=0}^{\infty}y_k\right\}$$

$$\leq 1-\alpha,$$

since $0\leq x_0\leq 1$. Therefore, $f\in\overline{co}\left(\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\right)$.

Conversely, suppose that f, which is of form (2), belongs to $\overline{co}\left(\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\right)$. Set

$$x_k=\frac{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}{(1-\alpha)\,p^{n+2}}\left|a_{k+p}\right|\quad (k=1,2,\ldots),$$

$$y_k=\frac{\left(p^2+\mu\lambda k\right)\left(p+\lambda k\right)^n}{(1-\alpha)\,p^{n+2}}\left|b_{k+p}\right|\quad (k=0,1,2,\ldots)$$

and

$$x_0=1-\sum_{k=1}^{\infty}x_k-\sum_{k=1}^{\infty}y_k.$$

Then we have

$$f(z)=\sum_{k=0}^{\infty}x_kh_k(z)+y_kg_k(z),$$

as required.

Remark 2. Theorem 3 means that the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ is convex and closed, thus we have

$$\overline{co}\left(\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\right)\equiv\mathcal{NHS}_p(n,\lambda,\mu;\alpha).$$

The following theorem gives the bounds for modulus of functions in $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$.

Theorem 4. Let $p,n\in\mathbb{N},\lambda\geq 1,0\leq\mu\leq p$ and $0\leq\alpha<1$. If $f\in\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, then

$$\left|f(z)\right|\leq\left(1+\left|b_p\right|\right)r^p+\frac{1-\left|b_p\right|-\alpha}{\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n}r^{p+1}$$

and

$$\left|f(z)\right|\geq\left(1-\left|b_p\right|\right)r^p-\frac{1-\left|b_p\right|-\alpha}{\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n}r^{p+1},$$

for $|z|=r<1$.

Proof. We note that

$$\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \geq \left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n, \tag{10}$$

for all $k \in \mathbb{N}$. Since $f \in \mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, by Theorem 2, we have

$$|b_p| + \sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \left(|a_{k+p}|+|b_{k+p}|\right) \leq 1-\alpha.$$

Hence, by the inequality (10), we have

$$\sum_{k=1}^{\infty}\left(|a_{k+p}|+|a_{k+p}|\right) \leq \frac{1-|b_p|-\alpha}{\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n}.$$

Now, taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &\leq \left(1+|b_p|\right)r^p + \sum_{k=1}^{\infty}\left(|a_{k+p}|+|b_{k+p}|\right)r^{k+p} \\
&\leq \left(1+|b_p|\right)r^p + \sum_{k=1}^{\infty} r^{p+1}\left(|a_{k+p}|+|b_{k+p}|\right) \\
&\leq \left(1+|b_p|\right)r^p + \frac{1-|b_p|-\alpha}{\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n} r^{p+1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|f(z)| &\geq \left(1-|b_p|\right)r^p - \sum_{k=1}^{\infty}\left(|a_{k+p}|+|b_{k+p}|\right)r^{k+p} \\
&\geq \left(1-|b_p|\right)r^p - \sum_{k=1}^{\infty} r^{p+1}\left(|a_{k+p}|+|b_{k+p}|\right) \\
&\geq \left(1-|b_p|\right)r^p - \frac{1-|b_p|-\alpha}{\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n} r^{p+1}.
\end{aligned}$$

For given $|b_p| \leq 1-\alpha$, the function

$$f_0(z) = z^p - |b_p|\overline{z}^p - \frac{1-|b_p|-\alpha}{\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n}\overline{z}^{p+1}$$

shows that the lower bound given in Theorem 4 is sharp.

Theorem 4 yields a covering result for the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ as follows:

Corollary 1 Let $p, n \in \mathbb{N}, \lambda \geq 1, 0 \leq \mu \leq p$ and $0 \leq \alpha < 1$. If $f \in \mathcal{NHS}_p(n, \lambda, \mu; \alpha)$ then

$$\left\{ w : |w| < 1 - |b_p| - \frac{1 - |b_p| - \alpha}{\left(1 + \frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n} \right\} \subset f(\mathbb{D}).$$

3 Neighborhood properties for the class $\mathcal{NHS}_p(n, \lambda, \mu; \alpha)$. In this section, we study some properties of neighborhood for the class $\mathcal{NHS}_p(n, \lambda, \mu; \alpha)$. Now, we define the $(\mu, \lambda) - \delta$ neighborhood of $f \in \mathcal{H}_p$ by

$$N_\delta^{p,\mu,\lambda}(f) = \left\{ F \in \mathcal{H}_p : F(z) = z^p - \sum_{k=1}^{\infty} |A_{k+p}| z^{k+p} - \sum_{k=0}^{\infty} |B_{k+p}| \overline{z}^{k+p} \right.$$

$$\left. \text{and } \sum_{k=0}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)\left(|a_{k+p} - A_{k+p}| + |b_{k+p} - B_{k+p}|\right) \leq \delta \right\}.$$

Theorem 5. Let $p, n \in \mathbb{N}, \lambda \geq 1, 0 \leq \mu \leq p$ and $0 \leq \alpha < 1$ and let $f \in \mathcal{NHS}_p(n, \lambda, \mu; \alpha)$. If

$$\delta \leq \left(1 - \left(\frac{p+1}{p}\right)^{1-n}\right)\left(1 - |b_p| - \alpha\right), \tag{11}$$

then $N_\delta^{p,\mu,\lambda}(f) \subset \mathcal{NHS}_p(n, \lambda, \mu; \alpha)$.

Proof. Let

$$F(z) = z^p - \sum_{k=1}^{\infty} |A_{k+p}| z^{k+p} - \sum_{k=0}^{\infty} |B_{k+p}| \overline{z}^{k+p}$$

belong to $N_\delta^{p,\mu,\lambda}(f)$. Now,

$$|B_p| + \sum_{k=1}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)\left(|A_{k+p}| + |B_{k+p}|\right)$$

$$\leq |B_p - b_p| + |b_p| + \sum_{k=1}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)\left(|a_{k+p} - A_{k+p}| + |b_{k+p} - B_{k+p}|\right)$$

$$+ \sum_{k=1}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)\left(|a_{k+p}| + |b_{k+p}|\right)$$

$$\leq \delta + |b_p| + \left(\frac{p+1}{p}\right)^{1-n} \sum_{k=1}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \left(|a_{k+p}| + |b_{k+p}|\right)$$

$$\leq \delta + |b_p| + \left(\frac{p+1}{p}\right)^{1-n}\left(1-|b_p|-\alpha\right).$$

Hence, if δ satisfies

$$\delta + |b_p| + \left(\frac{p+1}{p}\right)^{1-n}\left(1-|b_p|-\alpha\right) \leq 1-\alpha, \tag{12}$$

then F belong to the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ and the inequality (12) is equivalent to (11), hence the proof of Theorem 5 is completed.

4 Inclusion Relationships Involving the Class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$.

In this section, we obtain some inclusion relationships involving the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$. Now, we denote by $\mathcal{HS}_p^*(\alpha)$ the class of harmonic starlike functions of order α. And $\mathcal{HS}_p^*(0) \equiv \mathcal{HS}_p^*$. The following lemma is a suffcient condition for which f belongs to $\mathcal{HS}_p^*(\alpha)$.

Lemma 6 [1, Ahuja and Jahangiri] Let $p \in \mathbb{N}$ and $0 \leq \alpha < 1$. And let $f = h + \overline{g}$ be given by (1). If f satisfies the following inequality:

$$\sum_{k=1}^{\infty} \frac{k+p(1-\alpha)}{p-\alpha}|a_{k+p}| + \sum_{k=0}^{\infty} \frac{k+p(1-\alpha)}{p-\alpha}|b_{k+p}| \leq 2 - \frac{p(1-\alpha)}{p-\alpha}.$$

Then $f \in \mathcal{HS}_p^*(\alpha)$.

And, using Theorem 2, we can easily obtain the following results.

Theorem 7. Let $p,n \in \mathbb{N}, 0 \leq \alpha < 1, \lambda \geq 1$ and $0 \leq \mu \leq p$, and $n_i \in \mathbb{N}$, $0 \leq \alpha_i < 1, \lambda_i \geq 1$ and $0 \leq \mu_i \leq p$, for i=1,2. Then, the following four inclusion relationships hold:

(1) $\mathcal{NHS}_p(n,\lambda,\mu;\alpha_2) \subset \mathcal{NHS}_p(n,\lambda,\mu;\alpha_1)$, for $\alpha_1 \leq \alpha_2$.

(2) $\mathcal{NHS}_p(n_2,\lambda,\mu;\alpha) \subset \mathcal{NHS}_p(n_1,\lambda,\mu;\alpha)$, for $n_1 \leq n_2$.

(3) $\mathcal{NHS}_p(n,\lambda_2,\mu;\alpha) \subset \mathcal{NHS}_p(n_1,\lambda_1,\mu;\alpha)$, for $\lambda_1 \leq \lambda_2$.

(4) $\mathcal{NHS}_p(n,\lambda,\mu_2;\alpha) \subset \mathcal{NHS}_p(n,\lambda,\mu_1;\alpha)$, for $\mu_1 \leq \mu_2$.

We can easily obtain from Theorem 2 and Lemma 6 that $\mathcal{NHS}_p(1,1,0;0) \subset \mathcal{HS}_p^*$. Hence, by Theorem 7, we have

$$\mathcal{NHS}_p(n,\lambda,\mu;\alpha) \subset \mathcal{NHS}_p(1,1,0;0) \subset \mathcal{HS}_p^*.$$

for all n,α,λ and $u(n \in \mathbb{N}, 0 \leq \alpha < 1, \lambda \geq 1$ and $0 \leq \mu \leq p)$. Therefore, we can obtain the following result.

Theorem 8. Let $n \in \mathbb{N}, 0 \leq \alpha < 1, \lambda \geq 1$ and $0 \leq \mu \leq p$. Each function belonging to $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$ maps the unit disk onto a starlike domain.

Next, we obtain the order of starlikeness for functions in the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$

Theorem 9. Let $p,n\in\mathbb{N}, 0\le\alpha<1, \lambda\ge1$ and $0\le\mu\le p$. Then, the following inclusion relationship holds:

$$\mathcal{NHS}_p(n,\lambda,\mu;\alpha)\subset\mathcal{HS}_p^*(\beta),$$

where

$$\beta=\frac{(1-\alpha)p^n-p}{p(2\alpha-1)-2}.$$

Proof. Let f be of from (2) and belong the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$. Since f satisfies the inequality (8), we have

$$\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n\left|b_{k+p}\right|\le1-\alpha.$$

Furthermore, this inequality gives us that

$$\sum_{k=1}^{\infty}\left|b_{k+p}\right|\le\frac{1-\alpha-\left|b_p\right|}{A}, \tag{13}$$

where

$$A=\left(1+\frac{\mu\lambda}{p^2}\right)\left(\frac{p+\lambda}{p}\right)^n. \tag{14}$$

Since $k+p(1-\beta)\le(p+\lambda k)^n$ for all $k\in\mathbb{N}$, we have

$$\sum_{k=1}^{\infty}\frac{k+p(1-\beta)}{p-\beta}\left|a_{k+p}\right|+\sum_{k=1}^{\infty}\frac{k+p(1+\beta)}{p-\beta}\left|b_{k+p}\right|$$

$$\le\sum_{k=1}^{\infty}\frac{(p+\lambda k)^n}{p-\beta}\left|a_{k+p}\right|+\sum_{k=0}^{\infty}\frac{(p+\lambda k)^n}{p-\beta}\left|b_{k+p}\right|+\frac{2p\beta}{p-\beta}\sum_{k=0}^{\infty}\left|b_{k+p}\right|$$

$$=\sum_{k=1}^{\infty}\frac{(p+\lambda k)^n}{p-\beta}\left(\left|a_{k+p}\right|+\left|b_{k+p}\right|\right)+\frac{2p\beta}{p-\beta}\sum_{k=0}^{\infty}\left|b_{k+p}\right|+\frac{p^n+2p\beta}{p-\beta}\left|b_p\right|. \tag{15}$$

It follows from (13), (15) and the inequality $1+(\mu\lambda k)/p^2\ge1\,(k\in\mathbb{N})$ that

$$\sum_{k=1}^{\infty}\frac{k+p(1-\beta)}{p-\beta}\left|a_{k+p}\right|+\sum_{k=1}^{\infty}\frac{k+p(1+\beta)}{p-\beta}\left|b_{k+p}\right|$$

$$\le\frac{p^n}{p-\beta}\sum_{k=0}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n\left(\left|a_{k+p}\right|+\left|b_{k+p}\right|\right)+\frac{2p\beta}{p-\beta}\frac{1-\alpha-\left|b_p\right|}{A}+\frac{p^n+2p\beta}{p-\beta}\left|b_p\right|, \tag{16}$$

where A is given by (13). Since $f\in\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, applying Theorem 2 to the inequality (16), we get

$$\sum_{k=1}^{\infty}\frac{k+p(1-\beta)}{p-\beta}|a_{k+p}|+\sum_{k=0}^{\infty}\frac{k+p(1+\beta)}{p-\beta}|b_{k+p}|$$
$$\leq\frac{p^n}{p-\beta}\left(1-\alpha-|b_p|\right)+\frac{2p\beta}{p-\beta}\frac{1-\alpha-|b_p|}{A}+\frac{p^n+2p\beta}{p-\beta}|b_p|$$
$$=\frac{(1-\alpha)\left(p^n+2p\beta\right)}{p-\beta}=2-\frac{p(1-\beta)}{p-\beta},$$

where A is given by (13). Hence, by Lemma 6, we have that f is in the class $\mathcal{HS}_p^*(\beta)$.

5 Partial Sums of The Integral Operator. Let f_1 and f_2 be analytic functions defined in $\mathbb{D}$ with the following expressions:

$$f_1(z)=\sum_{k=0}^{\infty}a_k z^k \text{ and } f_2(z)=\sum_{k=0}^{\infty}b_k z^k.$$

Then the convolution (or Hadamard product) of $f_1(z)$ and $f_2(z)$, denoted by $(f_1*f_2)(z)$, is defined by

$$(f_1*f_2)(z)=f_1(z)=\sum_{k=0}^{\infty}a_k b_k z^k\ (z\in\mathbb{D}).$$

Similarly, for two harmonic functions $f_1=h_1+\overline{g}_1$ and $f_2=h_2+\overline{g}_2$, the convolution of f_1 and f_2 is defined by $(f_1*f_2)(z)=(h_1*h_2)(z)+\overline{(g_1*g_2)(z)}$.

To derive our results of this section, we shall require the following lemmas.

Lemma 10. [6, Gasper] If $-1<\alpha\leq A=4.5678018...$, then

$$\frac{1}{1+\alpha}+\sum_{k=1}^{n}\frac{1}{k+\alpha}\prod_{j=1}^{m}\cos(k\theta_j)\geq 0\ (n\in\mathbb{N}).$$

Lemma 11. [7, Jahangiri and Farahmand] Let $P(z)$ be analytic in $\mathbb{D}$, $P(0)=1$ and $\operatorname{Re}\{P(z)\}>1/2$ in $\mathbb{D}$. For functions F analytic in $\mathbb{D}$, the convolution $P*F$ takes values in the convex hull of the image on $\mathbb{D}$ under F.

Now, let us put $F(z)=\mathcal{L}_p(f;z)$ for $f\in\mathcal{H}_p$, where $\mathcal{L}_p$ is the integral operator defined by (3). And let F_j be the j-th partial sum of the function F.

Theorem 12. Let $p\in\{1,2,3\}, n\in\mathbb{N}, \lambda\geq 1, 0\leq\mu\leq p$ and $0\leq\alpha<1$. If f is of the form (1) with b_p=0 and $f\in\mathcal{HS}_p(n,\lambda,\mu;\alpha)$, then $F_j\in\mathcal{HS}_p(n,\lambda,\mu;\alpha)$, where

$$\beta = 1 - \frac{2(1-\alpha)(p+1)}{p+2}. \tag{49}$$

Proof. First of all, we shall prove that

$$\operatorname{Re}\left(\sum_{k=1}^{j} \frac{z^k}{k+p+1}\right) \geq -\frac{1}{p+2} \tag{17}$$

in $\mathbb{D}$. For $0 \leq r < 1$ and for $0 \leq |\theta| \leq \pi$, write $z = re^{i\theta}$. By the minimum principle for harmonic functions, we have

$$\operatorname{Re}\left(\sum_{k=1}^{j} \frac{z^k}{k+p+1}\right) \geq \sum_{k=1}^{j} \frac{(-1)^k}{k+p+1}.$$

Since $\alpha = 1 + p \leq A$, it follows from Lemma 10 that

$$\frac{1}{p+2} + \sum_{k=1}^{j} \frac{(-1)^k}{k+p+1} \geq 0$$

Therefore, the inequality (17) holds in $\mathbb{D}$. Now, let f be of the form (1) and belong to $\mathcal{HS}_p(n, \lambda, \mu; \alpha)$. Since

$$\operatorname{Re}\left\{\frac{\Omega^n_{\lambda,p,\mu}(h;z)}{z^p} + \overline{\left(\frac{\Omega^n_{\lambda,p,\mu}(g;z)}{z^p}\right)}\right\} > \alpha \, (z \in \mathbb{D}),$$

we have

$$\operatorname{Re}\left\{1 + \frac{1}{2(1-\alpha)} \sum_{k=1}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \left(a_{k+p} + b_{k+p}\right) z^k\right\} > \frac{1}{2} (z \in \mathbb{D}). \tag{18}$$

for $F_j = H_j + \overline{G_j}$, we have

$$\begin{aligned}
&\left(\Omega^n_{\lambda,p,\mu}(H_j;z) + \Omega^n_{\lambda,p,\mu}(G_j;z)\right)/z^p \\
&= 1 + \sum_{k=1}^{j} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \frac{p+1}{k+p+1} \left(a_{k+p} + b_{k+p}\right) z^k \\
&= P(z) * Q(z)
\end{aligned}$$

where

$$P(z) = 1 + \frac{1}{2(1-\alpha)} \sum_{k=1}^{\infty} \left(1 + \frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \left(a_{k+p} + b_{k+p}\right) z^k$$

and

$$Q(z) = 1 + 2(1-\alpha)(p+1) \sum_{k=1}^{j} \frac{z^k}{k+p+1}.$$

By (15), we have $\operatorname{Re}\big(P(z)\big) > 1/2$. And, it follows from the condition $p \in \{1,2,3\}$ and the inequality (17) that

$$\operatorname{Re}\big(Q(z)\big) = 1 + 2(1-\alpha)(p+1)\operatorname{Re}\left(\sum_{k=1}^{j} \frac{z^k}{k+p+1}\right) > 1 - 2(1-\alpha)\frac{p+1}{p+2}.$$

Hence, by Lemma 11, we have

$$\operatorname{Re}\left\{\frac{\Omega^n_{\lambda,p,\mu}\left(H_j;z\right)}{z^p} + \overline{\left(\frac{\Omega^n_{\lambda,p,\mu}\left(G_j;z\right)}{z^p}\right)}\right\} = \operatorname{Re}\left\{P(z) * Q(z)\right\} > 1 - 2(1-\alpha)\frac{p+1}{p+2} = \beta.$$

Theorem 13. Let f be given by (2) with $b_p=0$ and let $f \in \mathcal{NHS}_p(n,\lambda,\mu;\alpha)$. Then the function $F(z)$ belongs to $\mathcal{NHS}_p(n,\lambda,\mu;\rho)$, where

$$\rho = 1 - \frac{(1-\alpha)(p+1)}{p+2}.$$

The result is sharp. And, the converse need not be true.

Proof. Let f be given by (2) with $b_p=0$ and let $f \in \mathcal{NHS}_p(n,\lambda,\mu;\alpha)$. Then, by Theorem 2 with $b_p=0$, we have

$$\sum_{k=1}^{\infty} \frac{1}{1-\alpha}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \left(\left|a_{k+p}\right| + \left|b_{k+p}\right|\right) \le 1.$$

and

$$F(z) = z^p - \sum_{k=1}^{\infty} \frac{p+1}{k+p+1}\left|a_{k+p}\right| z^{k+p} - \sum_{k=1}^{\infty} \frac{p+1}{k+p+1}\left|b_{k+p}\right| \overline{z}^{k+p}.$$

Using Theorem 2 again, if

$$\sum_{k=1}^{\infty} \frac{1}{1-\sigma}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n \frac{p+1}{k+p+1}\left(\left|a_{k+p}\right| + \left|b_{k+p}\right|\right) \le 1, \tag{19}$$

then $F \in \mathcal{NHS}_p(n,\lambda,\mu;\sigma)$. Furthermore, if

$$\sigma \le 1 - \frac{(1-\alpha)(p+1)}{k+p+1} := \rho_k,$$

for all k =1,2,3,..., then (19) holds. Since ρ_k is an increasing function on k we have

$$\rho := \inf_{k \in \mathbb{N}} \rho_k = \rho_1 = -\frac{(1-\alpha)(p+1)}{p+2}. \tag{20}$$

Consequently, we have $F \in \mathcal{NHS}_p(n,\lambda,\mu;\rho)$, where ρ is given by (20). To show the sharpness, we consider the function $f(z) \in \mathcal{H}_p$ given by

Consequently, we have $F \in \mathcal{NHS}_p(n,\lambda,\mu;\rho)$, where ρ is given by (20). To show the sharpness, we consider the function $f(z) \in \mathcal{H}_p$ given by

$$f(z) = h(z) + \overline{g(z)} = z^p - \frac{(1-\alpha)}{\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n}|x|z^{p+1} - \frac{(1-\alpha)}{\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n}|y|\overline{z}^{p+1},$$

where $|x|+|y|=1$. Then, we have

$$F(z) = H(z) + \overline{G(z)} = z^p - \frac{p+1}{p+2}\frac{(1-\alpha)}{\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n}|x|z^{p+1} - \frac{p+1}{p+2}\frac{(1-\alpha)}{\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n}|y|\overline{z}^{p+1}.$$

And

$$\left(\Omega^n_{\lambda,p,\mu}(H;z) + \Omega^n_{\lambda,p,\mu}(G;z)\right)/z^p = 1 - \frac{(1-\alpha)(p+1)}{p+2}z \to 1 - \frac{(1-\alpha)(p+1)}{p+2},$$

for $z \to 1$. Hence, the result is sharp. To show that the converse of the above theorem need not to be true, we consider the function

$$F(z) = H(z) + \overline{G(z)} = z^p - \frac{(1-\sigma)}{\left(1+\frac{2\mu\lambda}{p^2}\right)\left(\frac{p+2\lambda}{p}\right)^n}|x|z^{p+2} - \frac{(1-\sigma)}{\left(1+\frac{2\mu\lambda}{p^2}\right)\left(\frac{p+2\lambda}{p}\right)^n}|y|\overline{z}^{p+2},$$

where $|x|+|y|=1$ and $\sigma = 1-(1-\alpha)(p+1)/(p+2)$. Then, Theorem 2 guarantees that $F(z) \in \mathcal{NHS}_p(n,\lambda,\mu;\sigma)$. But the corresponding function

$$f(z) = \frac{1}{p+1}(zH(z))' + \overline{\frac{1}{p+1}(zG(z))'}$$

$$= z^p - \frac{p+3}{p+1}\frac{(1-\sigma)}{\left(1+\frac{2\mu\lambda}{p^2}\right)\left(\frac{p+2\lambda}{p}\right)^n}|x|z^{p+2} - \frac{p+3}{p+1}\frac{(1-\sigma)}{\left(1+\frac{2\mu\lambda}{p^2}\right)\left(\frac{p+2\lambda}{p}\right)^n}|y|\overline{z}^{p+2}$$

does not belong to the class $\mathcal{NHS}_p(n,\lambda,\mu;\alpha)$, since

$$\sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n|a_{p+k}| + \sum_{k=1}^{\infty}\left(1+\frac{\mu\lambda k}{p^2}\right)\left(\frac{p+\lambda k}{p}\right)^n|b_{p+k}| = \frac{p+3}{p+2}(1-\alpha) > 1-\alpha.$$

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

BAILEY TYPE TRANSFORMS AND APPLICATIONS

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(Received : October 14, 2015; Final form : November 22, 2015)

ABSTRACT

The aim of this paper is to establish new series transforms of Bailey type and to show that these Bailey type transforms work as efficiently as the classical one and give not only new q-hypergeometric identities, converting double or triple series into a very-well-poised ${}_{10}\phi_9$ or ${}_{12}\phi_{11}$ series but can also be utilized to derive new double and triple series Rogers-Ramanujan type identities and corresponding infinite families of Rogers-Ramanujan type identities.

2010 Mathematics Subject Classification : 33D15, 33D70, 05A19

Keywords : Bailey transform, Bailey Lemma, Bailey pairs, q-hypergeometric series, Rogers-Ramanujna type identities.

1 Introduction. It is well-known that W. N. Bailey [23, 24] established a series transform known as Bailey's transform and gave a mechanism to derive ordinary and q-hypergeometric identities and Rogers-Ramanujan type identities. Using Bailey's transform, Slater [61, 62] derived 130 Rogers-Ramanujan type identities. It was Andrews [3, 4, 10] and [9], who exploited the Bailey's transform in the form of Bailey pair and Bailey chains to show that all of the 130 identities given by Slater [61, 62, 63] can be embedded in infinite families of multiple series Rogers-Ramanujan type identities.

After Bailey [23, 24] and notably after Andrews [9], a large number of Mathematicians have worked on Bailey's transform in the form of Bailey lemma and Bailey chain to make applications in the theory of generalized hypergeometric series, number theory, partition theory, combinatorics, physics and computer-algebra (see[1,2,5,6,7,8,11,12,13,14,15,16,17,18,19, 20,21,22,25,26,28,29,31,32,33, 34,35,36,38,41,42,43,49,50,51,52,53,54,55,56, 57,58,59,60,64,66,67,68,74,73,70,69, 72] and [71]).

A survey of all the above mentioned papers reveals that the work done by Bailey [23,24] and Andrews [9] may be considered as basic and fundamental in this field. At this point there arises one natural question,

"Can there be other series transforms of Bailey type, which not only produce new ordinary and q-hypergeometric identities and Rogers-Ramanujan type identities but can also be iterated to provide multi sum versions of the Rogers-Ramanujan type identities?"

This paper provides the answer in affirmative.

In this paper, first we establish two new series transform of Bailey type in Section 2 as theorems 2.1 and 2.2 and then using these theorems we derive five new q-hypergeometric identities converting double or triple q-hypergeometric series into a very-well-poised ${}_{10}\phi_9$ or ${}_{12}\phi_{11}$ series in Section 3. In Section 4, we establish the first Bailey type lemma and using it derive four new double series Rogers-Ramanujan type identities of modulo 7 and 5 and embed these in multiple series Rogers-Ramanujan type identities of modulo $4k+3$ and $4k+1$. In Section 5, we establish second Bailey type lemma and then using it derive two new triple series Rogers-Ramanujan type identities of modulo 9 and embed these in multiple series Rogers-Ramanujan type identities of modulo $6s+3$. During the presentation of both lemmas in Section 4 and 5, we establish only, the main results corresponding to Andrews [9, 10] presentation of Bailey lemma.

It may be noted that the double and triple series Rogers-Ramanujan type identities presented in this paper do not arise as merely "one or two level up" in the standard Bailey chain from well-known single sum identities. Furthermore, the original Rogers-Ramanujan identities (see [39,

45, 46]) were also of modulus 5 but were single sum identities. A modulo 7 identity with single sum is by Rogers [47, Eq. 6; p. 331] and as a one level up is by Andrews [4, Eq. (1.8); p. 4083]. Andrews [4] obtained his identity of modulo 7 by particularizing the general multi sum odd moduli identities to double series case. After comparing our identities with the results of references [34,35,36,55,56,57,58,66,67] and so forth, we can say that the identities derived in this paper are new and most attractive and symmetric among all the double and triple series Rogers-Ramanujan type identities derived previously.

The main tools in developing these transforms are two series rearrangements (2.5) and (2.10). Some of the instances of use of (2.5) and (2.10) may be found in the works of Burchnall and Chaundy [27], Shankar [48] and Srivastava and Manocha [65, p.335]. The notation for double q-Kampe de Feriet in the eqn (3.1) and (3.3) is from [65, Eq. (282), (283); p. 349] and of a triple series (3) in eqn (3.5) is simply a q-analogue of a particular case of the Srivastava's general triple series [65, Eq. (14), (15); p. 44]. In view of notational difficulties mentioned in [65, pp. 270-274], the double non q-Kampé de Fériet series in eqns (3.2) and (3.4) have been written explicitly without using any notation. However, notation of q-analogue of Srivastava-Daoust's series [65, Eq. 284; p.350] may be used in these cases. Remaining definitions and notations are from [33, 10] .

2. Bailey Type Transforms

2.1. First Bailey Type Transform

Theorem 2.1. If $\beta_{(n,l)} = \sum_{m=0}^{\min(n,l)} \alpha_m u_{n-m} u'_{l-m} v_{n+m} v'_{l+m} t_{n-l} w_{l+n}$ (2.1)

and

$$\gamma_m = \sum_{n=m}^{\infty}\sum_{l=m}^{\infty} \delta_n \delta'_l u_{n-m} u'_{l-m} v_{n+m} v'_{l+m} t_{n-1} w_{l+n} \tag{2.2}$$

then, subject to convergence conditions

$$\sum_{m=0}^{\infty} \alpha_m \gamma_m = \sum_{n,l=0}^{\infty} \beta_{(n,l)} \delta_n \delta'_l, \tag{2.3}$$

where, $\alpha_r, \delta_r, u_r, \delta'_r, u'_r, v'_r, t_r$ and w_r are any functions of r only.

Proof. Observe that

$$\sum_{m=0}^{\infty}\alpha_m\gamma_m=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{l=m}^{\infty}\alpha_m\delta_n\delta'_l u_{n-m}u'_{l-m}u_{n+m}v'_{l+m}t_{n-1}w_{l+n}. \tag{2.4}$$

If this double series is convergent, then using [30; p.10, lemma 3], viz.,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}A(m,n,l)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\min(n,l)}A(m,n-m,l-m), \tag{2.5}$$

in (2.4) after the replacement of n by $n+m$ and of l by $l+m$, we get

$$\sum_{m=0}^{\infty}\alpha_m\gamma_m=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\min(n,l)}\alpha_m\delta_n\delta'_l u_{n-m}u'_{l-m}u_{n+m}v'_{l+m}t_{n-1}w_{l+n}=\sum_{n,l=0}^{\infty}\beta_{(n,l)}\delta_n\delta'_l.$$

2.2 Second Bailey Type Transform.

Theorem 2.2

If $$\beta(n,l,k)=\sum_{m=0}^{\min(n,l,k)}\alpha_m u_{n-m}u'_{l-m}v''_{k-m}v_{n+m}v'_{l+m}v''_{k+m} \tag{2.6}$$

and

$$\gamma_m=\sum_{n=m}^{\infty}\sum_{l=m}^{\infty}\sum_{k=m}^{\infty}\delta_n\delta'_l\delta''_k u_{n-m}u'_{l-m}u''_{k-m}u_{n+m}v'_{l+m}u''_{k+m} \tag{2.7}$$

then, subject to convergence conditions

$$\sum_{m=0}^{\infty}\alpha_m\gamma_m=\sum_{n,l,k=0}^{\infty}\beta_{(n,l,k)}\delta_n\delta'_l\delta''_k \tag{2.8}$$

where $\alpha_r,\delta_r,u_r,v_r,\delta'_r,u'_r,v'_r,\delta''_r,u''_r$ and v''_r are any functions of r only.

Proof. Observe that

$$\sum_{m=0}^{\infty}\alpha_m\gamma_m=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{l=m}^{\infty}\sum_{k=m}^{\infty}\alpha_m,\delta_n,\delta'_l\delta''_k u_{n-m}u'_{l-m}u''_{k-m}v_{n+m}v'_{l+m}u''_{k+m}. \tag{2.9}$$

If this double series is convergent, then using [30; p.10, lemma 3], viz.,

$$\sum_{m=0}^{\infty}\sum_{n,l,k=0}^{\infty}A(m,n,l,k)\min(n,l,k)=\sum_{n,l,k=0}^{\min(n,l,k)}A(m,n-m,l-m,k-m), \tag{2.10}$$

in (2.8) after the replacement of n,l and k by $n+m,l+m$ and $k+m$ respectively, we

get

$$\sum_{m=0}^{\infty}\alpha_m\gamma_m=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\min.(n,l,k)}\alpha_m,\delta_n,\delta'_l\delta''_k u_{n-m}u'_{l-m}u''_{k-m}v_{n+m}v'_{l+m}v''_{k+m}$$

$$=\sum_{n,l,k=0}^{\infty}\beta_{(n,l,k)}\delta_n\delta'_l\delta''_k.$$

Here, it may be noted that in these two theorems various new sequences of other combinations of involved summation indices may be introduced in $\beta_{(n,l)}$ or $\beta_{(n,l,k)}$ and γ_m. But we have presented these theorems with only those sequences, which have been used at least once in this paper. Further, it may be possible in many cases to decide the new expressions for $\alpha_r,\delta_r,u_r,v_r,\delta'_r,u'_r,v'_r,\delta''_r,u''_r,v''_r,t_r$ and w_r, which yield closed forms for $\beta_{(m,l)}$ or $\beta_{(n,l,k)}$ and γ_m using one or more of the known summation theorems. Thus, many more new results may be discovered. But we have discussed certain cases which use the well-known classical summation theorems [33, Appendix, II. 5, 6, 12, 21, 22] only. It may be emphasized that there may be other various appropriate cases to have the closed forms for $\beta_{(n,l)}$ or $\beta_{(n,l,k)}$ and γ_m in Theorems 2.1 and 2.2.

Further, many series rearrangements, similar to (2.5) and (2.10), may also be utilized to discuss other Bailey type transforms and their applications.

3 New q-hypergeometric Identities. In the Bailey type transforms, discussed in the previous Section as Theorems 2.1 and 2.2, we observed five expressions for $\alpha_r,\delta_r,u_r,v_r,\delta'_r,u'_r,v'_r,\delta''_r,u''_r,v''_r,t_r$ and w_r, which yield closed forms for $\beta_{(n,l)}$ or $\beta_{(n,l,k)}$ and γ_m and lead to five new q-hypergeometric identities.

The identities are as follows:

$$\Phi^{1:3:2}_{1:2:2}\left[\begin{matrix}\frac{qa}{d}: & b,c,q^{-M}; & b',c',q^{-N} \\ qa & \frac{qa}{d},\frac{bcq^{-M}}{a}; & \frac{qa}{d},\frac{b'c'q^{-N}}{a};q,q\end{matrix}\right]$$

$$=\frac{\left(\frac{qa}{b},\frac{qa}{c};q\right)_M}{\left(qa,\frac{qa}{bc};q\right)_M}\frac{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_N}{\left(qa,\frac{qa}{b'c'};q\right)_N}\,{}_{10}W_9\left(a;d,b,c,b',c',q^{-M},q^{-N};q,\frac{a^3q^{3+M+N}}{bcb'c'd}\right). \quad (3.1)$$

$$\sum_{n,l=0}^{\infty}\frac{\left(\frac{A}{a};q\right)_{n-l}}{\left(\frac{qa}{B};q\right)_{n-l}}\frac{\left(\frac{AB}{a};q\right)_{n+l}}{(qa;q)_{n+l}}\frac{\left(A,q\sqrt{A},-q\sqrt{A},b,c,\frac{Aaq^{1+M}}{bc},\frac{qa}{B},q^{-M};q\right)_n}{\left(\sqrt{A},-\sqrt{A},\frac{qA}{b},\frac{qA}{c},\frac{bcq^{-M}}{a},\frac{AB}{a},Aq^{1+M};q\right)_n}$$

$$\frac{\left(B,q\sqrt{B},-q\sqrt{B},b',c',\frac{Baq^{1+N}}{b'c'},\frac{qa}{A},q^{-N};q\right)_l}{\left(\sqrt{B},-\sqrt{B},\frac{qB}{b'},\frac{qB}{c'},\frac{b'c'q^{-N}}{a},\frac{AB}{a},Bq^{1+N};q\right)_l}\frac{q^n\left(\frac{qAB}{a^2}\right)^l}{(q;q)_n(q;q)_l}$$

$$=\frac{\left(qA,\frac{qa}{b},\frac{qa}{c},\frac{qA}{bc};q\right)_M}{\left(qa,\frac{qA}{b},\frac{qA}{c},\frac{qa}{bc};q\right)_M}\frac{\left(qB,\frac{qa}{b'},\frac{qa}{c'},\frac{qB}{b'c'};q\right)_N}{\left(qa,\frac{qB}{b'},\frac{qB}{c'},\frac{qa}{b'c'};q\right)_N}$$

$${}_{12}W_{11}\left(a;\frac{qa^2}{AB},b,c,b',c',\frac{Aaq^{1+M}}{bc},\frac{Baq^{1+N}}{b'c'},q^{-M},q^{-N};q,q\right), \quad (3.2)$$

$$\Phi^{2:2;8}_{2,1,7}\left[\begin{matrix}b',\frac{qa}{d} & : & c',q^{-M}; & \sqrt{qb'},-\sqrt{qb'},b,c,\frac{ab'q^{1+N}}{bc},\frac{b'c'}{a},\frac{b'q^{-M}}{a},q^{-N} \\ qa,\frac{b'c'q^{-M}}{a} & : & \frac{qa}{d}; & \sqrt{b'},-\sqrt{b'},\frac{qb'}{b},\frac{qb'}{c},\frac{bcq^{-N}}{a},b'q^{1+N},\frac{qa}{d}\end{matrix};q,q\right]$$

$$=\frac{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_M}{\left(qa,\frac{qa}{b'c'};q\right)_M}\frac{\left(qb',\frac{qa}{b},\frac{qa}{c},\frac{qb'}{bc};q\right)_N}{\left(qa,\frac{qb'}{b},\frac{qb'}{c},\frac{qa}{bc};q\right)_N}$$

$${}_{10}W_9\left(a;d,b,c,c',\frac{q^{1+N}ab'}{bc},q^{-M},q^{-N};q,\frac{q^{2+M}a^2}{b'c'}\right), \quad (3.3)$$

$$\sum_{n,l=0}^{\infty}\frac{\left(\frac{b'}{A};q\right)_{n-l}}{\left(\frac{qa}{A};q\right)_{n-l}}\frac{(b';q)_{n-1}}{(qa;q)_{n-l}}$$

$$\frac{\left(q\sqrt{b'}-q\sqrt{b'},b,c,\frac{b'q^{-M}}{a},\frac{b'c'}{a},\frac{b'aq^{1+N}}{bc},\frac{b'c'q^{-M}}{A},\frac{qa}{A},q^{-N};q\right)_n}{\left(\sqrt{b'}-\sqrt{b'},\frac{b'q^{-M}}{A},\frac{b'c'}{A},\frac{qb'}{b},\frac{qb'}{c},\frac{b'c'q^{-M}}{a},\frac{bcq^{-N}}{a},b'q^{1+N};q\right)_n}$$

$$\frac{\left(q\sqrt{A},-q\sqrt{A},c',A,\frac{Aaq^{1+M}}{b'c'},q^{-M};q\right)_l}{\left(\sqrt{A},-\sqrt{A},\frac{qA}{c'},\frac{b'c'q^{-M}}{a},Aq^{1+M};q\right)_l}\frac{q^n\left(\frac{q^{2+M}a^2}{b'c'}\right)^l}{(q;q)_n(q;q)_l}$$

$$=\frac{\left(qA,\frac{qa}{b'},\frac{qa}{c'},\frac{qA}{b'c'};q\right)_M}{\left(qa,\frac{qA}{b'},\frac{qA}{c'},\frac{qA}{b'c'};q\right)_M}\frac{\left(qb',\frac{qa}{b},\frac{qa}{c},\frac{qb'}{bc};q\right)_N}{\left(qa,\frac{qb'}{b},\frac{qb'}{c},\frac{qa}{bc};q\right)_N}$$

$$ {}_{10}W_9\left(a;b,c,c',\frac{a^2q^{1+N}}{b'c'},\frac{ab'q^{1+N}}{bc},q^{-M},q^{-N};q,\frac{qc'}{aq^M}\right) \tag{3.4}$$

and

$$\Phi^{(3)}\left[\begin{matrix} qa:: & : & : & : & b,c,q^{-N}; & b',c',q^{-L}; & b'',c'',q^{-K} \\ :: & qa: & qa: & qa: & \frac{bcq^{-N}}{a}; & \frac{b'c'q^{-L}}{a} & \frac{b'',c'',q^{-K}}{a};q;q,q,q \end{matrix}\right]$$

$$=\frac{\left(\frac{qa}{b},\frac{qa}{c};q\right)_N}{\left(qa,\frac{qa}{bc};q\right)_N}\frac{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_L}{\left(qa,\frac{qa}{b'c'};q\right)_L}\frac{\left(\frac{qa}{b''},\frac{qa}{c''};q\right)_K}{\left(qa,\frac{qa}{b''c''};q\right)_K}$$

$$ {}_{12}W_{11}\left(a;b,c,b',c',b'',c'',q^{-N},q^{-L},q^{-K};q,\frac{a^4q^{N+L+K+4}}{bcb'c'b''c''}\right). \tag{3.5}$$

Proof of (3.1). For the choice

$$\alpha_r=\frac{\left(a,q\sqrt{a},-q\sqrt{a},d;q\right)_r q^{r^2-r}\left(\frac{aq^3}{d}\right)^r}{\left(\sqrt{a},-\sqrt{a},\frac{qa}{d},q;q\right)_r},\delta_r=\frac{\left(b,c,q^{-M};q\right)_r}{\left(\frac{bcq^{-M}}{a};q\right)_r}$$

$$\delta'_r = \frac{\left(b',c',d^{-N};q\right)_r}{\left(\frac{b'c'q^{-N}}{a};q\right)_r}, \upsilon_r = \upsilon'_r = \frac{1}{(qa;q)_r}, u_r = u'_r = \frac{q^r}{(q;q)_r},$$

and $t_r = w_r = 1$ in (2.1) and (2.2), the q-Pfaff-Saalchutz sum [33, p. 237, (II. 12) and the terminating very-well-poised ${}_6\Phi_5$ sum [33, p. 238, (II. 21)] can be used to have

$$\beta_{(n,l)} = \frac{\left(\frac{qa}{d};q\right)_{n+l} q^{n+l}}{(qa;q)_{n+l}\left(\frac{qa}{d},q;q\right)_n\left(\frac{qa}{d},q;q\right)_l}$$

and

$$\gamma_m = \frac{\left(\frac{qa}{b},\frac{qa}{c};q\right)_M}{\left(qa,\frac{qa}{bc};q\right)_M}\frac{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_N}{\left(qa,\frac{qa}{b'c'};q\right)_N}\frac{\left(b,c,b',c',q^{-M},q^{-N};q\right)_m q^{-m^2+m}}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{b'},\frac{qa}{c'},aq^{1+M},aq^{1+N};q\right)_m}\left(\frac{a^2q^{M+N}}{bcb'c'}\right)^M.$$

Substituting these values in (2.3), we obtain the result (3.1).

Proof of (3.2). For the choice

$$\alpha_r = \frac{\left(a,q\sqrt{a},-q\sqrt{a},\frac{qa^2}{AB};q\right)_r\left(\frac{qAB}{a^2}\right)}{\left(\sqrt{a},-\sqrt{a},\frac{AB}{a},q;q\right)_r}$$

$$\delta_r = \frac{\left(q\sqrt{A},-q\sqrt{A},b,c\frac{Aaq^{1+M}}{bc};q^{-M};q\right)_r}{\left(\sqrt{A},-\sqrt{A},\frac{qA}{b},\frac{qA}{c},\frac{bcq^{-M}}{a},Aq^{1+M};q\right)_r},$$

$$\delta'_r = \frac{\left(q\sqrt{B},-q\sqrt{B},b',c',\frac{Baq^{1+N}}{b'c'};q^{-N};q\right)_r}{\left(\sqrt{B},-\sqrt{B},\frac{qB}{b'},\frac{qB}{c'},\frac{b'c'q^{-N}}{a};Bq^{1+N};q\right)_r}$$

$$v_r=\frac{(A;q)_r}{(qa;q)_r}, v'_r=\frac{(B;q)_r}{(qa;q)_r}, u_r=\frac{\left(\frac{A}{a};q\right)_r q^r}{(q;q)_r}, u'_r=\frac{\left(\frac{B}{a};q\right)_r q^r}{(q;q)_r},$$

and $t_r=w_r=1$ in (2.1) and (2.2) and using the Jackson's q-analogue of Dougall's ${}_7F_6$ sum [33, p. 238, (II. 22)] we can obtain

$$\beta_{(n,l)}=\frac{\left(\frac{A}{a};q\right)_{n-l}\left(\frac{AB}{a};q\right)_{n+l}\left(A,\frac{qa}{B};q\right)_n\left(B,\frac{qa}{A};q\right)_l}{\left(\frac{qa}{B};q\right)_{n-l}(qa;q)_{n+l}\left(\frac{AB}{a},q;q\right)_n\left(\frac{AB}{a},q;q\right)_l}q^n\left(\frac{AB}{a^2}\right)^l$$

and

$$\gamma_m=\frac{\left(qA,\frac{qa}{b},\frac{qa}{c},\frac{qA}{bc};q\right)_M}{\left(qa,\frac{qA}{b},\frac{qA}{c},\frac{qa}{bc};q\right)_M}\frac{\left(qB,\frac{qa}{b'},\frac{qa}{c'},\frac{qB}{b'c'};q\right)_N}{\left(qa,\frac{qB}{b'},\frac{qB}{c'},\frac{qa}{b'c'};q\right)_N}$$

$$\frac{\left(b,c,\frac{Aaq^{1+M}}{bc},q^{-M},b',c',\frac{Baq^{1+N}}{b'c'},q^{-N};q\right)_m}{\left(\frac{qa}{b},\frac{qa}{c},\frac{bcq^{-M}}{A},aq^{1+M},\frac{qa}{b'},\frac{qa}{c'},\frac{b'c'q^{-N}}{B},aq^{1+N};q\right)_m}\left(\frac{a^2}{AB}\right)^m$$

Substituting these values in (2.3), we obtain the result (3.2).

Proof of (3.3). For the choice

$$\alpha_r=\frac{\left(a,q\sqrt{a},-q\sqrt{a},d;q\right)_r q^{r^2}\left(\frac{q^2a}{d}\right)^r}{\left(\sqrt{a},-\sqrt{a},\frac{qa}{d},q;q\right)_r}$$

$$\delta_r=\frac{\left(q\sqrt{b'},-q\sqrt{b'},b,c,\frac{ab'q^{1+N}}{bc},q^{-N},\frac{b'c'}{a},\frac{b'q^{-M}}{a};q\right)_r}{\left(\sqrt{b'},-\sqrt{b'},\frac{qb'}{b},\frac{qb'}{c},\frac{bcq^{-N}}{a},b'q^{1+N};q\right)_r},$$

$$v_r=\frac{1}{(qa;q)_r}=v'_r, u_r=\frac{q^r}{(q;q)_r}=u'_r, w_r=\frac{(b';q)_r}{\left(\frac{b'c'q^{-M}}{a};q\right)},$$

$\delta'_r = \left(c', q^{-M}; q\right)_r$ and $t_r=1$ in (2.1) and (2.2), the terminating very-well-poised ${}_6\Phi_5$ sum [33,p.238,(II.21)], the q-Pfaff-Saalschutz sum [33, p. 237, (II. 12)] and the Jackson's q-analogue of Dougall's 7F6 sum [33, p. 238, (II. 22)] leads to

$$\beta_{(n,l)} = \frac{\left(b', \frac{qa}{d}; q\right)_{n+l}}{\left(aq, \frac{b'c'q^{-M}}{a}; q\right)_{n+l}} \beta_{(n,l)} = \frac{\left(b', \frac{qa}{d}; q\right)_{n+l}}{\left(aq, \frac{b'c'q^{-M}}{a}; q\right)_{n+l}} \frac{q^{n+l}}{\left(\frac{qa}{d}, q; q\right)_n \left(\frac{qa}{d}, q; q\right)_l}$$

and

$$\gamma_m = \frac{\left(\frac{qa}{b'}, \frac{qa}{c'}; q\right)_M}{\left(qa, \frac{qa}{b'c'}; q\right)_M} \frac{\left(ab', \frac{qa}{b}, \frac{qa}{c}, \frac{qb'}{bc}; q\right)_N}{\left(qa, \frac{qb'}{b}, \frac{qb'}{c}, \frac{qa}{bc}; q\right)_N}$$

$$\frac{\left(b, c, c', \frac{ab'q^{1+N}}{bc}, q^{-M}, q^{-N}; q\right)_m}{\left(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{c'}, \frac{bcq^{-N}}{b'}, aq^{1+M}, aq^{1+N}; q\right)_m} \left(\frac{a}{b'c'}\right)^m q^{-m^2+mM}.$$

Substituting these values in (2.3), we obtain the result (3.3).

Proof of (3.4). For the choice

$$\alpha_r = \frac{\left(a, q\sqrt{a}, -q\sqrt{a}, \frac{a^2q^{1+M}}{b'c'}; q\right)_r \left(\frac{ab'c'}{a^2q^M}\right)^r}{\left(\sqrt{a}, -\sqrt{a}, \frac{b'c'q^{-M}}{a}, q; q\right)_r},$$

$$\delta_r = \frac{\left(q\sqrt{b'}, -q\sqrt{b'}, b, c, \frac{ab'q^{1+N}}{bc}, q^{-N}, \frac{b'c'}{a}, \frac{b'q^{-M}}{a}; q\right)_r}{\left(\sqrt{b'}, -\sqrt{b'}, \frac{qb'}{b}, \frac{qb'}{c}, \frac{bcq^{-N}}{a}, b'q^{1+N}; q\right)_r},$$

$$\delta'_r = \frac{\left(q\sqrt{A}, -q\sqrt{A}, c', q^{-M}; q\right)_r}{\left(\sqrt{A}, -\sqrt{A}, \frac{qA}{c'}, Aq^{1+M}; q\right)_r}, w_r = \frac{(b'; q)_r}{\left(\frac{b'c'q^{-M}}{a}; q\right)_r}, t_r = \frac{\left(\frac{b'}{A}; q\right)_r}{\left(\frac{b'c'q^{-M}}{Aa}; q\right)_r},$$

$$v_r = \frac{\left(\frac{b'c'q^{-M}}{A};q\right)_r}{(qa;q)_r}, v'_r = \frac{(A;q)_r}{(qa;q)_r}, u_r = \frac{\left(\frac{b'c'q^{-M}}{Aa};q\right)_r q^r}{(q;q)_r} \text{ and } u'_r = \frac{\left(\frac{A}{a};q\right)_r q^r}{(q;q)_r},$$

in (2.1) and (2.2), the Jackson's q-analogue of Dougall's ${}_7F_6$ sum [33, p. 238, (II. 22)] gives rise to

$$\beta_{(n,l)} = \frac{\left(\frac{a}{A};q\right)_{n-l}(b':q)_{n+l}\left(\frac{b'c'q^{-M}}{A},\frac{qa}{A};q\right)_n\left(A,\frac{aAq^{1+M}}{b'c'};q\right)_l}{\left(\frac{qa}{A};q\right)_{n-l}(qa;q)_{n+l}\left(\frac{b'c'q^{-M}}{A},q;q\right)_n\left(\frac{b'c'q^{-M}}{A},q;q\right)_l} q^n\left(\frac{b'c'q^{-M}}{a^2}\right)^l$$

and

$$\gamma_m = \frac{\left(qA,\frac{qa}{b'},\frac{qa}{c'},\frac{qA}{b'c'};q\right)_M}{\left(qa,\frac{qA}{b'},\frac{qA}{c'},\frac{qa}{b'c'};q\right)_M}\frac{\left(qb',\frac{qa}{b},\frac{qa}{c},\frac{qb'}{bc};q\right)_N}{\left(qa,\frac{qb'}{b},\frac{qb'}{c},\frac{qa}{bc};q\right)_N}$$

$$\frac{\left(b,c,c',\frac{b'aq^{1+N}}{bc},q^{-M},q^{-N};q\right)_m}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{c'},\frac{bcq^{-N}}{b'},aq^{1+M},aq^{1+N};q\right)_m}\left(\frac{a}{b'}\right)^m.$$

Substituting these values in (2.3), we obtain the result (3.4).

Proof of (3.5). For the choice

$$\alpha_r = \frac{\left(a,q\sqrt{a},-q\sqrt{a};q\right)_r a^r q^{4r} q^{3\binom{r}{2}}}{\left(\sqrt{a},-\sqrt{a},q;q\right)_r}, \delta_r = \frac{\left(b,c,q^{-N};q\right)_r}{\left(\frac{bcq^{-N}}{a};q\right)_r},$$

$$\delta'_r = \frac{\left(b',c',q^{-L};q\right)_r}{\left(\frac{b'c'q^{-L}}{a};q\right)_r}, \delta^n_r = \frac{\left(b'',c'',q^{-K};q\right)_r}{\left(\frac{b''c''q^{-K}}{a};q\right)_r}.$$

$$v_r = v'_r = v''_r = \frac{1}{(qa;q)_r}, \text{ and } u_r = u'_r = u''_r = \frac{q^r}{(q;q)_r},$$

in (2.6) and (2.7), the q-Pfaff-Saalschütz sum [33, p. 237, (II. 12)] and the terminating very-well-poised ${}_6\Phi_5$ sum [33, p. 238, (II. 21)] readily yields

$$\beta_{(n,l,k)} = \frac{(qa;q)_{n+l+k}\, q^{n+l+k}}{(qa;q)_{n+l}(qa;q)_{n+k}(qa;q)_{l+k}(q;q)_n(q;q)_l(q;q)_k}$$

and

$$\gamma_m = \frac{\left(\frac{qa}{b},\frac{qa}{c};q\right)_N}{\left(qa,\frac{qa}{bc};q\right)_N}\frac{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_L}{\left(qa,\frac{qa}{b'c'};q\right)_L}\frac{\left(\frac{qa}{b''},\frac{qa}{c''};q\right)_K}{\left(qa,\frac{qa}{b''c''};q\right)_K}$$

$$\frac{\left(b,c,b',c',b'',c'',q^{-N},q^{-L},q^{-K};q\right)_m(-1)^m q^{-3\binom{m}{2}}}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{b'},\frac{qa}{c'},\frac{qa}{b''},\frac{qa}{c''},aq^{1+N},aq^{1+L},aq^{1+K};q\right)_m}\left(\frac{a^3q^{N+L+K}}{bcb'c'b''c''}\right)^m.$$

Substituting these values in (2.8), we obtain the result (3.5).

4. First Bailey type lemma and applications to Rogers-Ramanujan type identities

4.1. First Bailey Type Lemma or FBTL.

Theorem 4.1. If for $n,l \geq 0$

$$\beta_{(n,l)} = \sum_{m=0}^{\min(n,l)} \frac{\alpha_m}{(q;q)_{n-m}(q;q)_{l-m}(aq;q)_{n+m}(aq;q)_{l+m}}, \tag{4.1}$$

then

$$\beta'_{(n,l)} = \sum_{m=0}^{\min(n,l)} \frac{\alpha'_m}{(q;q)_{n-m}(q;q)_{l-m}(aq;q)_{n+m}(aq;q)_{l+m}}, \tag{4.2}$$

where

$$\alpha'_m = \frac{(b,c,b',c';q)_m\left(\frac{a^2q^2}{bcb'c'}\right)^m \alpha_m}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{b'},\frac{qa}{c'};q\right)_m} \tag{4.3}$$

and

$$\beta'_{(M,N)} = \sum_{n,l=0}^{\infty} \frac{(b,c;q)_n\left(\frac{qa}{bc};q\right)_{M-n}\left(\frac{qa}{bc}\right)^n}{\left(\frac{qa}{b},\frac{qa}{c};q\right)_M(q;q)_{M-n}} \frac{(b',c';q)_l\left(\frac{qa}{b'c'};q\right)_{N-l}\left(\frac{qa}{b'c'}\right)^l \beta_{(n,l)}}{\left(\frac{qa}{b},\frac{qa}{c};q\right)_N(q;q)_{N-l}}, \tag{4.4}$$

Remark. A pair of sequences $\left(\alpha_m, \beta_{(n,l)}\right)$ related by (4.1) may be called as a "first Bailey type pair" or "FBTP" and the Theorem 4.1 may be rephrased as: If $\left(\alpha_m, \beta_{(n,l)}\right)$ is a FBTP, so is $\left(\alpha'_m, \beta'_{(n,l)}\right)$ where this new FBTP is given by (4.3) and (4.4).

Proof. To prove FBTL, we apply Bailey type transform (2.3) with the choice:

$$\delta_r = \frac{\left(b, c, q^{-M}; q\right)_r q^r}{\left(\frac{bcq^{-M}}{a}; q\right)_r}, \delta'_r = \frac{\left(b', c', q^{-N}; q\right)_r q^r}{\left(\frac{b'c'q^{-N}}{a}; q\right)_r}, u_r = v'_r = \frac{1}{(qa;q)_r}, u_r = u'_r = \frac{1}{(q;q)_r},$$

and $t_r = w_r = 1$ then as in the proof of (3.1), we get

$$\gamma_m = \frac{\left(\frac{qa}{b}, \frac{qa}{c}; q\right)_M}{\left(qa, \frac{qa}{bc}; q\right)_M} \frac{\left(\frac{qa}{b'}, \frac{qa}{c'}; q\right)_N}{\left(qa, \frac{qa}{b'c'}; q\right)_N} \frac{\left(b, c, b', q^{-M}, q^{-N}; q\right)_m q^{-m^2+m}}{\left(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{b'} \frac{qa}{c'}, aq^{1+M}, aq^{1+N}; q\right)_m} \left(\frac{a^2 q^{2+M+N}}{bcb'c'}\right)^m.$$

Now we can prove (4.2) as shown below:

$$\sum_{m=0}^{\min(M,N)} \frac{\alpha'_m}{(q;q)_{M-m}(q;q)_{N-m}(aq;q)_{M+m}(aq;q)_{N+m}}$$

$$= \sum_{m=0}^{\min(M,N)} \frac{(b,c,b',c';q)_m \left(\frac{a^2q^2}{bcb'c'}\right)^m \alpha_m}{\left(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{b'}, \frac{qa}{c'}, aq^{1+M}, aq^{1+N}; q\right)_m (q;q)_{M-m}(q;q)_{N-m}(aq:q')_{M+m}(aq;q)_{N+m}}$$

$$= \frac{1}{(qa,q;q)_M} \frac{1}{(qa,q;q)_N}$$

$$\sum_{m=0}^{\min(M,N)} \frac{\left(b,c,b',c',q^{-M},q^{-N};q\right)_m q^{-m^2+m}}{\left(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{b'}, \frac{qa}{c'}, aq^{1+M}, aq^{1+N}; q\right)_m (q;q)_{M-m}(q;q)_{N-m}(aq;q)_{M+m}(aq;q)_{N+}}$$

$$= \frac{\left(\frac{qa}{bc}; q\right)_M}{\left(\frac{qa}{b}, \frac{qa}{c}, q; q\right)_M} \frac{\left(\frac{qa}{b'c'}; q\right)_N}{\left(\frac{qa}{b'}, \frac{qa}{c'}, q; q\right)_N} \sum_{m=0}^{\min.(M,N)} \gamma_m \alpha_m$$

$$=\frac{\left(\frac{qa}{bc};q\right)_M}{\left(\frac{qa}{b},\frac{qa}{c},q;q\right)_M}\frac{\left(\frac{qa}{b'c'};q\right)_N}{\left(\frac{qa}{b'},\frac{qa}{c'},q;q\right)_N}\sum_{n=0}^{M}\sum_{l=0}^{N}\beta_{(n,l)}\delta_n\delta_l' \qquad \text{(by Theorem 4.1)}$$

$$=\frac{\left(\frac{qa}{bc};q\right)_M}{\left(\frac{qa}{b},\frac{qa}{c},q;q\right)_M}\frac{\left(\frac{qa}{b'c'};q\right)_N}{\left(\frac{qa}{b'},\frac{qa}{c'},q;q\right)_N}\sum_{n=0}^{M}\sum_{l=0}^{N}\frac{\left(b,c,q^{-M};q\right)_n q^n\left(b',c',q^{-N};q\right)_l q^l\beta_{(n,l)}}{\left(\frac{bcq^{-M}}{a};q\right)_n\left(\frac{b'c'q^{-N}}{a};q\right)_l}$$

$$=\beta'_{(M,N)}.$$

The last line follows by manipulating q-shifted factorials, which reduces the previous expression to (4.4).

Now to derive Rogers-Ramanujan type identities, we need the following result obtained by substituting the values of α'_m and $\beta'_{(n,l)}$ from (4.3) and (4.4) into (4.2), viz.

$$\sum_{n,l=0}^{\infty}\frac{(b,c;q)_n\left(\frac{qa}{bc};q\right)_{M-n}\left(\frac{qa}{bc}\right)^n}{\left(\frac{qa}{b},\frac{qa}{c};q\right)_M (q;q)_{M-n}}\frac{(b',c';q)_l\left(\frac{qa}{b'c'};q\right)_{N-l}\left(\frac{qa}{b'c'}\right)^l\beta_{(n,l)}}{\left(\frac{qa}{b},\frac{qa}{c};q\right)_N (q;q)_{N-l}}$$

$$=\sum_{m=0}^{\min(M,N)}\frac{(b,c,b',c';q)_m\left(\frac{a^2q^2}{bcb'c'}\right)^m\alpha_m}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{b'},\frac{qa}{c'};q\right)_m (q;q)_{M-m}(q;q)_{N-m}(aq;q)_{M+m}(aq;q)_{N+m}}. \tag{4.5}$$

Now taking $b,c,b',M,N\to\infty$ in (4.5), we get

$$\sum_{n,l=0}^{\infty}a^{n+l}q^{n^2+l^2}\beta_{(n,l)}=\frac{1}{(aq;q)_\infty^2}\sum_{m=0}^{\infty}a^{2m}q^{2m^2}\alpha_m, \tag{4.6}$$

for any FBTP $\left(\alpha_m,\beta_{(n,l)}\right)$. Now, we shall make use of the following two "FBTPs" deduced by us :

$$\alpha_m=\frac{(a;q)_m\left(1-aq^{2m}\right)}{(q;q)_m(1-a)}(-1)^m a^m q^{1/2\left(3m^2-m\right)},\ \beta_{(n,l)}=\frac{1}{(qa;q)_{n+l}(q;q)_n(q;q)_l}, \tag{4.7}$$

and

$$\alpha_m = \frac{(a;q)_m\left(1-aq^{2m}\right)}{(q;q)_m(1-a)}(-1)^m q^{1/2\left(m^2-m\right)},\ \beta_{(n,l)} = \frac{q^{nl}}{(qa;q)_{n+l}(q;q)_n(q;q)_l}. \tag{4.8}$$

The fact that "FBTPs" $\left(\alpha_m, \beta_{n,l}\right)$ given by (4.7) and (4.8) satisfy (4.1) may be verified by substituting them in (4.1) and then appealing to terminating very-well-poised ${}_6\phi_5$ sum [33, p. 238, (II. 21)] and taking $d \to \infty$ or $d \to 0$.

4.2. New double series Rogers-Ramanujan type identities and corresponding infinite families. To derive Rogers-Ramanujan type identities, we insert the "FBTPs" (4.7) and (4.8) in (4.6) to get (4.9) and (4.10), respectively, as given below:

$$\sum_{n,l=0}^{\infty} \frac{a^{n+l}q^{n^2+l^2}}{(qa;q)_{n+l}(q;q)_n(q;q)_l}$$

$$= \frac{1}{(aq;q)_\infty^2}\sum_{m=0}^{\infty}\frac{(a;q)_m\left(1-aq^{2m}\right)}{(q;q)_m(1-a)}(-1)^m a^{3m} q^{1/2\left(7m^2-m\right)}, \tag{4.9}$$

$$\sum_{n,l=0}^{\infty} \frac{a^{n+l}q^{n^2+l^2+nl}}{(qa;q)_{n+l}(q;q)_n(q;q)_l}$$

$$= \frac{1}{(aq;q)_\infty^2}\sum_{m=0}^{\infty}\frac{(a;q)_m\left(1-aq^{2m}\right)}{(q;q)_m(1-a)}(-1)^m q^{2m} q^{1/2\left(5m^2-m\right)}, \tag{4.10}$$

and then setting $a=1$ and $a=q$ in (4.9) and (4.10) and using Jacobi triple product identity, we obtain following four new double series Rogers-Ramanujan type identities:

$$\sum_{n,l=0}^{\infty}\frac{q^{n^2+l^2}}{(q;q)_{n+l}(q;q)_n(q;q)_l} = \frac{1}{(q;q)_\infty}\prod_{m=1,\not\equiv 0,\pm 3 (\mathrm{mod}\, 7)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.11}$$

$$\sum_{n,l=0}^{\infty}\frac{q^{n^2+l^2+n+l}}{(q^2;q)_{n+l}(q;q)_n(q;q)_l} = \frac{1}{(q^2;q)_\infty}\prod_{m=1,m\not\equiv 0,\pm 1 (\mathrm{mod}\, 7)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.12}$$

$$\sum_{n,l=0}^{\infty}\frac{q^{n^2+l^2+nl}}{(q;q)_{n+l}(q;q)_n(q;q)_l} = \frac{1}{(q;q)_\infty}\prod_{m=1,m\not\equiv 0,\pm 2 (\mathrm{mod}\, 5)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.13}$$

and

$$\sum_{n,l=0}^{\infty}\frac{q^{n^2+l^2+n+l+nl}}{\left(q^2;q\right)_{n+l}(q;q)_n(q;q)_l}=\frac{1}{\left(q^2;q\right)_{\infty}}\prod_{m=1,m\not\equiv 0,\pm1(\text{mod }5)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.14}$$

Further, like classical Bailey lemma, the idea of repeated application of FBTL may be expressed in the concept of the "first Bailey type chain." Given a FBTP $\left(\alpha_m,\beta_{(n,l)}\right)$ we can by FBTL produce new FBTP $\left(\alpha'_m,\beta'_{(n,l)}\right)$ defined by (4.3) and (4.4). From $\left(\alpha'_m,\beta'_{(n,l)}\right)$ we can create $\left(\alpha''_m,\beta''_{(n,l)}\right)$ merely by applying FBTL with $\left(\alpha'_m,\beta'_{(n,l)}\right)$ as initial FBTP. In this way we create a sequence of FBTPs:

$$\left(\alpha_m,\beta_{(n,l)}\right)\to\left(\alpha'_m,\beta'_{(n,l)}\right)\to\left(\alpha''_m,\beta''_{(n,l)}\right)\to\left(\alpha'''_m,\beta'''_{(n,l)}\right)\to\ldots;$$

and call it "first Bailey type chain". Using this chain we can establish:

Theorem 4.2

$$\sum_{m\geq 0}\frac{(b_1;q)_m(c_1;q)_m(b_2;q)_m(c_2;q)_m\ldots(b_k;q)_m(c_k;q)_m}{\left(\frac{aq}{b_1};q\right)_m\left(\frac{aq}{c_1};q\right)_m\left(\frac{aq}{b_2};q\right)_m\left(\frac{aq}{c_2};q\right)_m\ldots\left(\frac{aq}{b_k};q\right)_m\left(\frac{aq}{c_k};q\right)_m}$$

$$\frac{(b'_1;q)_m(c'_1;q)_m(b'_2;q)_m(c'_2;q)_m\ldots(b'_k;q)_m(c'_k;q)_m}{\left(\frac{aq}{b'_1};q\right)_m\left(\frac{aq}{c'_1};q\right)_m\left(\frac{aq}{b'_2};q\right)_m\left(\frac{aq}{c'_2};q\right)_m\ldots\left(\frac{aq}{b'_k};q\right)_m\left(\frac{aq}{c'_k};q\right)_m}$$

$$\frac{\left(q^{-M};q\right)_m\left(q^{-N};q\right)_m}{\left(aq^{1+M};q\right)_m\left(aq^{1+N};q\right)_m}\left(\frac{a^{2k}q^{2k+M+N}}{b_1c_1\ldots b_kc_kb'_1c'_1\ldots b'_kc'_k}\right)^m q^{-m^2+m}\alpha_m$$

$$=\frac{(aq;q)_M\left(\frac{aq}{b_kc_k};q\right)_M}{\left(\frac{aq}{b_k};q\right)_M\left(\frac{aq}{c_k};q\right)_M}\frac{(aq;q)_N\left(\frac{aq}{b'_kc'_k};q\right)_N}{\left(\frac{aq}{b'_k};q\right)_N\left(\frac{aq}{c'_k};q\right)_N}$$

$$\sum_{n_k\geq\ldots\geq n_1\geq 0,\,l_k\geq\ldots\geq l_1\geq 0}\frac{(b_k;q)_{n_k}(c_k;q)_{n_k}\ldots(b_1;q)_{n_1}(c_1;q)_{n_1}}{(q;q)_{n_k-n_{k-1}}(q;q)_{n_{k-1}-n_{k-2}}\ldots(q;q)_{n_2-n_1}}$$

$$\frac{(b_k';q)_{l_k}(c_k';q)_{l_k}\cdots(b_1';q)_{l_1}(c_1';q)_{l_1}}{(q;q)_{l_k-l_{k-1}}(q;q)_{l_{k-1}-l_{k-2}}\cdots(q;q)_{l_2-l_1}}\frac{(q^{-M};q)_{n_k}}{\left(\frac{b_kc_kq^{-M}}{a};q\right)_{n_k}}\frac{(q^{-N};q)_{l_k}}{\left(\frac{b_k'c_k'q^{-N}}{a};q\right)_{l_k}}$$

$$\frac{\left(\frac{aq}{b_{k-1}c_{k-1}};q\right)_{n_k-n_{k-1}}\cdots\left(\frac{aq}{b_1c_1};q\right)_{n_2-n_1}}{\left(\frac{aq}{b_{k-1}};q\right)_{n_k}\left(\frac{aq}{c_{k-1}};q\right)_{n_k}\cdots\left(\frac{aq}{b_1};q\right)_{n_2}\left(\frac{aq}{c_1};q\right)_{n_2}}$$

$$\frac{\left(\frac{aq}{b_{k-1}'c_{k-1}'};q\right)_{l_k-l_{k-1}}\cdots\left(\frac{aq}{b_1'c_1'};q\right)_{l_2-l_1}}{\left(\frac{aq}{b_{k-1}'};q\right)_{l_k}\left(\frac{aq}{c_{k-1}'};q\right)_{l_k}\cdots\left(\frac{aq}{b_1'};q\right)_{l_2}\left(\frac{aq}{c_1'};q\right)_{l_2}}$$

$$q^{n_1+n_2+\ldots+n_k+l_1+l_2+\ldots+l_k}a^{n_1+n_2+\ldots+n_{k-1}+l_1+l_2+\ldots+l_{k-1}}$$

$$(b_{k-1}c_{k-1})^{-n_{k-1}}\cdots(b_1c_1)^{-n_1}(b_{k-1}'c_{k-1}')^{-l_{k-1}}\cdots(b_1'c_1')^{-l_1}\beta_{(n_1,l_1)}. \tag{4.15}$$

Proof. Equation (4.15) is the result of a k-fold iteration of FBTL with the choice $b=b_i, c=c_i, b'=b_i', c'=c_i'$ at the i th step.

Theorem 4.3. If $\left(\alpha_m,\beta_{(n,l)}\right)$ is a FBTP (i.e. related by (4.1)), then

$$\sum_{n_k\ge\ldots\ge n_1\ge0,l_k\ge\ldots\ge l_1\ge0}\frac{a^{n_1+n_2+\ldots+n_k+l_1+l_2+\ldots+l_k}q^{n_1^2+n_2^2+\ldots+n_k^2+l_1^2+l_2^2+\ldots+l_k^2}\beta_{(n_1,l_1)}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{l_k-l_{k-1}}\cdots(q;q)_{l_2-l_1}}$$

$$=\frac{1}{(aq;q)_\infty^2}\sum_{m=0}^{\infty}q^{2km^2}a^{2km}\alpha_m. \tag{4.16}$$

Proof. Let $M,N,b_1,\ldots,b_k,c_1,\ldots c_k,b_1',\ldots,b_k',c_1',\ldots c_k'$ all tend to infinity in Theorem 4.2.

Theorem 4.3 may now be used to embed the double series Rogers-Ramanujan type identities (4.11)-(4.14) in an infinite family of such identities :

$$\sum_{n_k\ge\ldots\ge n_1\ge0,l_k\ge\ldots\ge l_1\ge0}\frac{q^{n_1^2+n_2^2+\ldots+n_k^2+l_1^2+l_2^2+\ldots+l_k^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{l_k-l_{k-1}}\cdots(q;q)_{l_2-l_1}}$$

$$\frac{1}{(q;q)_{n_1+l_1}(q;q)_{n_1}(q;q)_{l_1}}=\frac{1}{(q;q)_\infty}\prod_{m=1,m\not\equiv 0,\pm(2k+1)(\text{mod } 4k+3)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.17}$$

$$\sum_{n_k\geq\ldots\geq n_1\geq 0,l_k\geq\ldots l_1\geq 0}\frac{q^{n_1^2+n_2^2+\ldots+n_k^2+l_1^2+l_2^2+\ldots+l_k^2+n_1+n_2+\ldots+n_k+l_1+l_2+\ldots+l_k}}{(q;q)_{n_k-n_{k-1}}\ldots(q;q)_{n_2-n_1}(q;q)_{l_k-l_{k-1}}\ldots(q;q)_{l_2-l_1}}$$

$$\frac{1}{(q^2;q)_{n_1+l_1}(q;q)_{n_1}(q;q)_{l_1}}=\frac{1}{(q^2;q)_\infty}\prod_{m=1,m\not\equiv 0,\pm 1(\text{mod } 4k+3)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.18}$$

$$\sum_{n_k\geq\ldots\geq n_1\geq 0,l_k\geq\ldots l_1\geq 0}\frac{q^{n_1^2+n_2^2+\ldots+n_k^2+l_1^2+l_2^2+\ldots+l_k^2+n_1l_1}}{(q;q)_{n_k-n_{k-1}}\ldots(q;q)_{n_2-n_1}(q;q)_{l_k-l_{k-1}}\ldots(q;q)_{l_2-l_1}}$$

$$\frac{1}{(q;q)_{n_1+l_1}(q;q)_{n_1}(q;q)_{l_1}}=\frac{1}{(q;q)_\infty}\prod_{m=1,m\not\equiv 0,\pm 2(\text{mod } 4k+1)}^{\infty}\left(1-q^m\right)^{-1}, \tag{4.19}$$

and

$$\sum_{n_k\geq\ldots\geq n_1\geq 0,l_k\geq\ldots l_1\geq 0}\frac{q^{n_1^2+n_2^2+\ldots+n_k^2+l_1^2+l_2^2+\ldots+l_k^2+n_1+n_2+\ldots+n_k+l_1+l_2+\ldots+l_k+n_1l_1}}{(q;q)_{n_k-n_{k-1}}\ldots(q;q)_{n_2-n_1}(q;q)_{l_k-l_{k-1}}\ldots(q;q)_{l_2-l_1}}$$

$$\frac{1}{(q^2;q)_{n_1+l_1}(q;q)_{n_1}(q;q)_{l_1}}=\frac{1}{(q^2;q)_\infty}\prod_{m=1,m\not\equiv 0,\pm 1(\text{mod } 4k+1)}^{\infty}\left(1-q^m\right)^{-1}. \tag{4.20}$$

Equation (4.17) follows by using FBTP defined by (4.7) in (4.16) and then invoking to Jacobi triple product identity after taking $a=1$. For (4.18), we follow the same procedure with $a=q$. In the same way we establish (4.19) and (4.20) by using FBTP defined by (4.8).

5. Second Bailey type lemma and applications to Rogers Ramanujan type identities.

5.1 Second Bailey Type Lemma or SBTL.

Theorem 5.1. If for $n,l,k\geq 0$

$$\beta_{(n,l,k)}=\sum_{m=0}^{\min(n,l,k)}\frac{\alpha_m}{(q;q)_{n-m}(q;q)_{l-m}(q;q)_{k-m}(aq;q)_{n+m}(aq;q)_{l+m}(aq;q)_{k+m}}, \tag{5.1}$$

then

$$\beta'_{(n,l,k)}=\sum_{m=0}^{\min(n,l,k)}\frac{\alpha'_m}{(q;q)_{n-m}(q;q)_{l-m}(q;q)_{k-m}(aq;q)_{n+m}(aq;q)_{l+m}(aq;q)_{k+m}}, \tag{5.2}$$

where

$$\alpha'_m = \frac{(b,c,b',c',b'',c'';q)_m \left(\frac{a^3q^3}{bcb'c'b''c''}\right)^m \alpha_m}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{b'},\frac{qa}{c'}\frac{qa}{b''},\frac{qa}{c''};q\right)_m} \tag{5.3}$$

and

$$\beta'_{(N,L,K)} = \sum_{n,l,k=0}^{\infty} \frac{(b,c;q)_n \left(\frac{qa}{bc};q\right)_{N-n} \left(\frac{qa}{bc}\right)^n}{\left(\frac{qa}{b},\frac{qa}{c};q\right)_N (q;q)_{N-n}} \frac{(b',c';q)_l \left(\frac{qa}{b'c'};q\right)_{L-l} \left(\frac{qa}{b'c'}\right)^l}{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_L (q;q)_{L-l}}$$

$$\frac{(b'',c'';q)_k \left(\frac{qa}{b''c''};q\right)_{K-k} \left(\frac{qa}{b''c''}\right)^k \beta_{(n,l,k)}}{\left(\frac{qa}{b''},\frac{qa}{c''};q\right)_K (q;q)_{K-k}}. \tag{5.4}$$

Remark: A pair of sequences $\left(\alpha_m,\beta_{(n,l,k)}\right)$ related by (5.1) may be called as a "second Bailey type pair" or "SBTP" and the Theorem 5.1 may be rephrased as: If $\left(\alpha_m,\beta_{(n,l,k)}\right)$ is a SBTP, so is $\left(\alpha'_m,\beta'_{(n,l,k)}\right)$ where this new SBTP is given by (5.3) and (5.4).

Proof. To prove SBTL, we apply Bailey type transform (2.8) with the choice:

$$\delta_r = \frac{(b,c,q^{-N};q)_r q^r}{\left(\frac{bcq^{-N}}{a};q\right)_r}, \delta'_r = \frac{(b',c',q^{-L};q)_r q^r}{\left(\frac{b'c'q^{-L}}{a};q\right)_r}, \delta''_r = \frac{(b'',c'',q^{-K};q)_r q^r}{\left(\frac{b''c''q^{-K}}{a};q\right)_r},$$

and

$$v_r = v'_r = v''_r = \frac{1}{(qa;q)_r}, u_r = u'_r = u''_r = \frac{1}{(q;q)_r},$$

then following the proof of Theorem 4.1 we can also prove Theorem 5.1 .

Now to derive Rogers-Ramanujan type identities, we need the following result obtained by substituting the values of α'_m and $\beta'_{(n,l,k)}$ from (5.3) and (5.4) into (5.2), viz.,

$$\sum_{n,l,k=0}^{\infty} \frac{(b,c;q)_n \left(\frac{qa}{bc};q\right)_{N-n} \left(\frac{qa}{bc}\right)^n}{\left(\frac{qa}{b},\frac{qa}{c};q\right)_N (q;q)_{N-n}} \frac{(b',c';q)_l \left(\frac{qa}{b'c'};q\right)_{L-l} \left(\frac{qa}{b'c'}\right)^l}{\left(\frac{qa}{b'},\frac{qa}{c'};q\right)_L (q;q)_{L-l}}$$

$$\frac{(b'',c'';q)_k \left(\frac{qa}{b''c''};q\right)_{K-k} \left(\frac{qa}{b''c''}\right)^k \beta_{(n,l,k)}}{\left(\frac{qa}{b''},\frac{qa}{c''};q\right)_K (q;q)_{K-k}}$$

$$= \sum_{m=0}^{\min.(N,L,K)} \frac{(b,c,b',c',b'',c'';q)_m \left(\frac{a^3q^3}{bcb'c'b''c''}\right)^m}{\left(\frac{qa}{b},\frac{qa}{c},\frac{qa}{b'},\frac{qa}{c'}\frac{qa}{b''},\frac{qa}{c''};q\right)_m}$$

$$\frac{\alpha_m}{(q;q)_{N-m}(q;q)_{L-m}(q;q)_{K-m}(aq;q)_{N+m}(aq;q)_{L+m}(aq;q)_{K+m}}. \tag{5.5}$$

Now taking $b,c,b',c',b'',c'',M,N \to \infty$ in (4.5), we get

$$\sum_{n,l,k=0}^{\infty} a^{n+l+k} q^{n^2+l^2+k^2} \beta_{(n,l,k)} = \frac{1}{(aq;q)_\infty^3} \sum_{m=0}^{\infty} a^{3m} q^{3m^2} \alpha_m, \tag{5.6}$$

for any SBTP $\left(\alpha_m, \beta_{(n,l,k)}\right)$. Now, we shall make use of the following two "SBTP" deduced by us :

$$\alpha_m = \frac{(a;q)_m \left(1-aq^{2m}\right)}{(q;q)_m (1-a)} (-1)^m a^m q^{1/2\left(3m^2-m\right)},$$

$$\beta_{(n,l,k)} = \frac{(aq;q)_{n+l+k}}{(qa;q)_{n+l}(qa;q)_{n+k}(qa;q)_{l+k}(q;q)_n (q;q)_l (q;q)_k}.$$

The fact that "SBTP" $\left(\alpha_m, \beta_{(n,l,k)}\right)$ given by (5.7) satisfy (5.1) may be verified by substituting it in (5.1) and then appealing to terminating very-well-poised ${}_6\Phi_5$ sum [33, p.238, (II.21)].

5.2 New triple series Rogers-Ramanujan type identities and corresponding infinite families. To derive Rogers-Ramanujan type identities, we insert the "SBTP" (5.7) in (5.6) to get (5.8) as given below:

$$\sum_{n,l,k=0}^{\infty}\frac{a^{n+l+k}q^{n^2+l^2+k^2}(qa;q)_{n+l+k}}{(qa;q)_{n+l}(qa;q)_{n+k}(qa;q)_{l+k}(q;q)_n(q;q)_l(q;q)_k}$$

$$=\frac{1}{(aq;q)_\infty^3}\sum_{m=0}^{\infty}\frac{(a;q)_m\left(1-aq^{2m}\right)}{(q;q)_m(1-a)}(-1)^m a^{4m}q^{1/2\left(9m^2-m\right)} \tag{5.8}$$

and then setting $a=1$ and $a=q$ in (5.8) and using Jacobi triple product identity, we obtain following two new triple series Rogers-Ramanujan type identities:

$$\sum_{n,l,k=0}^{\infty}\frac{q^{n^2+l^2+k^2}(q;q)_{n+l+k}}{(q;q)_{n+l}(q;q)_{n+k}(q;q)_{l+k}(q;q)_n(q;q)_l(q;q)_l}$$

$$=\frac{1}{(q;q)_\infty^2}\prod_{\substack{m=1\\ m\not\equiv 0,\pm 4 \pmod 9}}^{\infty}\left(1-q^m\right)^{-1}, \tag{5.9}$$

$$\sum_{n,l,k=0}^{\infty}\frac{q^{n^2+l^2+k^2+n+l+k}\left(q^2;q\right)_{n+l+k}}{\left(q^2;q\right)_{n+l}\left(q^2;q\right)_{n+k}\left(q^2;q\right)_{l+k}(q;q)_n(q;q)_l(q;q)_k}$$

$$=\frac{1}{\left(q^2;q\right)_\infty^2}\prod_{\substack{m=1\\ m\not\equiv 0,\pm 1 \pmod 9}}^{\infty}\left(1-q^m\right)^{-1}. \tag{5.10}$$

Further, like classical Bailey lemma and FBTL, the idea of repeated application of SBTL may be expressed in the concept of the "second Bailey type chain" a sequence of SBTPs:

$$\left(\alpha_m,\beta_{(n,l,k)}\right)\to\left(\alpha'_m,\beta'_{(n,l,k)}\right)\to\left(\alpha''_m,\beta''_{(n,l,k)}\right)\to\left(\alpha'''_m,\beta'''_{(n,l,k)}\right)\to\dots .$$

Using this chain we can establish:

Theorem 5.2

$$\sum_{m\geq 0}\frac{(b_1;q)_m(c_1;q)_m(b_2;q)_m(c_2;q)_m\dots(b_s;q)_m(c_s;q)_m}{\left(\frac{aq}{b_1};q\right)_m\left(\frac{aq}{c_1};q\right)_m\left(\frac{aq}{b_2};q\right)_m\left(\frac{aq}{c_2};q\right)_m\dots\left(\frac{aq}{b_s};q\right)_m\left(\frac{aq}{c_s};q\right)_m}$$

$$\frac{(b'_1;q)_m(c'_1;q)_m(b'_2;q)_m(c'_2;q)_m\dots(b'_s;q)_m(c'_s;q)_m}{\left(\frac{aq}{b'_1};q\right)_m\left(\frac{aq}{c'_1};q\right)_m\left(\frac{aq}{b'_2};q\right)_m\left(\frac{aq}{c'_2};q\right)_m\dots\left(\frac{aq}{b'_s};q\right)_m\left(\frac{aq}{c'_s};q\right)_m}$$

$$\frac{(b_1'';q)_m (c_1'';q)_m (b_2'';q)_m (c_2'';q)_m \dots (b_s'';q)_m (c_s'';q)_m}{\left(\frac{aq}{b_1''};q\right)_m \left(\frac{aq}{c_1''};q\right)_m \left(\frac{aq}{b_2''};q\right)_m \left(\frac{aq}{c_2''};q\right)_m \dots \left(\frac{aq}{b_s''};q\right)_m \left(\frac{aq}{c_s''};q\right)_m}$$

$$\frac{(q^{-N};q)_m (q^{-L};q)_m (q^{-K};q)_m}{(aq^{1+N};q)_m (aq^{1+L};q)_m (aq^{1+K};q)_m} \left(\frac{a^{3s} q^{3s+N+L+K}}{b_1c_1\dots b_sc_sb_1'c_1'\dots b_s'c_s'b_1''c_1''\dots b_s''c_s''}\right)^m q^{3/2(-m^2+m)} \alpha_m$$

$$= \frac{(aq;q)_N \left(\frac{aq}{b_sc_s};q\right)_N}{\left(\frac{aq}{b_s};q\right)_N \left(\frac{aq}{c_s};q\right)_N} \frac{(aq;q)_L \left(\frac{aq}{b_s'c_s'};q\right)_L}{\left(\frac{aq}{b_s'};q\right)_L \left(\frac{aq}{c_s'};q\right)_L} \frac{(aq;q)_K \left(\frac{aq}{b_s''c_s''};q\right)_K}{\left(\frac{aq}{b_s''};q\right)_K \left(\frac{aq}{c_s''};q\right)_K}$$

$$\sum_{n_s \ge \dots \ge n_1 \ge 0, l_s \ge \dots \ge l_1 \ge 0, k_s \ge \dots \ge k_1 \ge 0} \frac{(b_s;q)_{n_s} (c_s;q)_{n_s} \dots (b_1;q)_{n_1} (c_1;q)_{n_1}}{(q;q)_{n_s-n_{s-1}} (q;q)_{n_{s-1}-n_{s-2}} \dots (q;q)_{n_2-n_1}}$$

$$\frac{(b_s';q)_{l_s} (c_s';q)_{l_s} \dots (b_1';q)_{l_1} (c_1';q)_{l_1}}{(q;q)_{l_s-l_{s-1}} (q;q)_{l_{s-1}-l_{s-2}} \dots (q;q)_{l_2-l_1}} \frac{(b_s'';q)_{k_s} (c_s'';q)_{k_s} \dots (b_1'';q)_{l_1} (c_1'';q) l_1}{(q;q)_{k_s-k_{s-1}} (q;q)_{k_{s-1}-k_{s-2}} \dots (q;q)_{k_2-k_1}}$$

$$\frac{(q^{-N};q)_{n_s}}{\left(\frac{b_sc_sq^{-N}}{a};q\right)_{n_s}} \frac{(q^{-L};q)_{l_s}}{\left(\frac{b_s'c_s'q^{-L}}{a};q\right)_{l_s}} \frac{(q^{-K};q)_{k_s}}{\left(\frac{b_s''c_s''q^{-K}}{a};q\right)_{k_s}} \frac{\left(\frac{aq}{b_{s-1}c_{s-1}};q\right)_{n_s-n_{s-1}} \dots \left(\frac{aq}{b_1c_1};q\right)_{n_2-n_1}}{\left(\frac{aq}{b_{s-1}};q\right)_{n_s} \left(\frac{aq}{c_{s-1}};q\right)_{n_s} \dots \left(\frac{aq}{b_1};q\right)_{n_2} \left(\frac{aq}{c_1};q\right)_{n_2}}$$

$$\frac{\left(\frac{aq}{b_{s-1}'c_{s-1}'};q\right)_{l_s-l_{s-1}} \dots \left(\frac{aq}{b_1'c_1'};q\right)_{l_2-l_1}}{\left(\frac{aq}{b_{s-1}'};q\right)_{l_s} \left(\frac{aq}{c_{s-1}'};q\right)_{l_s} \dots \left(\frac{aq}{b_1'};q\right)_{l_2} \left(\frac{aq}{c_1'};q\right)_{l_2}} \frac{\left(\frac{aq}{b_{s-1}''c_{s-1}''};q\right)_{k_s-k_{s-1}} \dots \left(\frac{aq}{b_1''c_1''};q\right)_{k_2-k_1}}{\left(\frac{aq}{b_{s-1}''};q\right)_{k_s} \left(\frac{aq}{c_{s-1}''};q\right)_{k_s} \dots \left(\frac{aq}{b_1''};q\right)_{k_2} \left(\frac{aq}{c_1''};q\right)_{k_2}}$$

$$q^{n_1+n_2+\dots+n_s+l_1+l_2+\dots+l_s+k_1+k_2+\dots+k_s} a^{n_1+n_2+\dots+n_{s-1}+l_1+l_2+\dots+l_{s-1}+k_1+k_2+\dots+k_{s-1}}$$

$$(b_{s-1}c_{s-1})^{-n_{s-1}} \dots (b_1c_1)^{-n_1} (b_{s-1}'c_{s-1}')^{-l_{s-1}} \dots (b_1'c_1')^{-l_1} (b_{s-1}'c_{s-1}')^{-k_{s-1}} \dots (b_1'c_1')^{-k_1} \beta_{(n_1,l_1,k_1)}. \quad (5.11)$$

Proof. Equation (5.11) is the result of a s-fold iteration of SBTL with the choice $b = b_i, c = c_i, b' = b_i', c' = c_i', b'' = b_i'', c'' = c_i''$ at the ith step.

Theorem 5.3. If $(\alpha_m, \beta_{(n,l,k)})$ is a SBTP (i.e. related by (5.1)), then

$$\sum_{n_s\ge\ldots\ge n_1\ge 0,\, l_s\ge\ldots\ge l_1\ge 0,\, k_s\ge\ldots\ge k_1\ge 0}\frac{q^{n_1^2+n_2^2+\ldots+n_s^2+l_1^2+l_2^2+l_2^2+\ldots+l_s^2+k_1^2+k_2^2+\ldots+k_s^2}}{(q;q)_{n_s-n_{s-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{l_s-l_{s-1}}\cdots(q;q)_{l_2-l_1}}$$

$$\frac{a^{n_1+n_2+\ldots+n_s+l_1+l_2+\ldots+l_s+k_1+k_2+\ldots+k_s}\beta_{(n_1,l_1,k_1)}}{(q;q)_{k_s-k_{s-1}}\cdots(q;q)_{k_2-k_1}}$$

$$=\frac{1}{(aq;q)_\infty^3}\sum_{m=0}^{\infty}q^{3sm^2}a^{3sm}\alpha_m. \tag{5.12}$$

Proof. Letting $N, L, K, b_1, \ldots, b_s, c_1, \ldots, c_s, b'_1, \ldots, b'_s, c'_1, \ldots, c'_s, b''_1, \ldots, b''_s, c''_1, \ldots, c''_s$ all tend to infinity in Theorem 5.2, the proof of Theorem 5.3 follows.

Theorem 5.3 may now be used to embed the triple series Rogers-Ramanujan type identities (5.9) and (5.10) in an infinite family of such identities:

$$\sum_{n_s\ge\ldots\ge n_1\ge 0,\, l_s\ge\ldots\ge l_1\ge 0,\, k_s\ge\ldots\ge k_1\ge 0}\frac{q^{n_1^2+\ldots+n_s^2+l_1^2+\ldots+l_s^2+k_1^2+\ldots+k_s^2}}{(q;q)_{n_s-n_{s-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{l_s-l_{s-1}}\cdots(q;q)_{l_2-l_1}}$$

$$\frac{(q;q)_{n_1+l_1+k_1}}{(q;q)_{k_s-k_{s-1}}\cdots(q;q)_{k_2-k_1}(q;q)_{n_1+l_1}(q;q)_{n_1+k_1}(q;q)_{l_1+k_1}}\,\frac{1}{(q;q)_{n_1}(q;q)_{l_1}(q;q)_{k_1}}$$

$$=\frac{1}{(q;q)_\infty^2}\prod_{m=1,\, m\not\equiv 0,\pm(3s+1)(\mathrm{mod}\, 6s+3)}^{\infty}\left(1-q^m\right)^{-1}. \tag{5.13}$$

and

$$\sum_{n_s\ge\ldots\ge n_1\ge 0,\, l_s\ge\ldots\ge l_1\ge 0,\, k_s\ge\ldots\ge k_1\ge 0}\frac{q^{n_1^2+\ldots+n_s^2+l_1^2+\ldots+l_s^2+k_1^2+\ldots+k_s^2}}{(q;q)_{n_s-n_{s-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{l_s-l_{s-1}}\cdots(q;q)_{l_2-l_1}}$$

$$\frac{q^{n_1+\ldots+n_s+l_1+\ldots+l_s+k_1+\ldots+k_s}}{(q;q)_{k_s-k_{s-1}}\cdots(q;q)_{k_2-k_1}(q^2;q)_{n_1+l_1}(q^2;q)_{n_1+k_1}(q^2;q)_{l_1+k_1}}\,\frac{(q^2;q)_{n_1+l_1+k_1}}{(q^2;q)_{n_1}(q^2;q)_{l_1}(q^2;q)_{k_1}}$$

$$=\frac{1}{(q^2;q)_\infty^2}\prod_{m=1,\, m\not\equiv 0,\pm(\mathrm{mod}\, 6s+3)}^{\infty}\left(1-q^m\right)^{-1}. \tag{5.14}$$

ACKNOWLEDGEMENT

The second author Dr. Vyas is thankful to the council of scientific and Industrial Research (CSIR), New Delhi, India for providing him Senior Research Fellowship.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

ON TWO VARIABLE GENERALIZED LUPAS OPERATORS AND THEIR APPLICATIONS

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(*Recieved : September* 07, 2015*; final form : October* 20, 2015)

ABSTRACT

In this work, we define a generalization of Lupas type operators for two variable functions and prove the Weierstrass approximation theorem for two variable functions in a rectangle. Then we analyze some of its special cases, properties, relations and applications in finding out the generating functions and Lupas- Durrmeyer type operators in two variables.

2010 Mathematics Subject Classification: 41A25, 41A30, 41A36, 33C20.

Keywords. Weierstrass approximation theorem for two variable functions, Lupas-Durrmeyer type operators, generating functions.

1. Introduction. Lupas proposed a family of linear positive operators mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $C[0,\infty)$ in the form (see Derriennic [1])

$$(L_n f)(x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right), \text{ where } x\in[0,\infty). \tag{1}$$

Here the continuous function f is given by $f:[a,b]\to\mathbb{R}; a,b\in\mathbb{R}$ and f be a uniform limit of polynomials on interval $[a,b]$. (See Weierstrass [9]).The continuity of the function f is that for any $\in>0$ there exists $\delta>0$ such that $|f(x)-f(y)|<\in/2$, for all $x,y\in\mathbb{R}$ with $|x-y|<\delta$. Again, Sahai and Prasad [7] presented a modified Lupas operators defined for functions integrable on $[0,\infty)$ given by

$$(M_n f)(x)=(n-1)\sum_{k=0}^{\infty}P_{n,k}(x)\int_0^{\infty}P_{n,k}(y)f(y)dy, P_{n,k}(t)=\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}} \quad (2)$$

Recently Kumar and Pathan [5] defined an operator $H_k f(x)=\frac{1}{k}\int_{-\infty}^{\infty}f(u)g\left(\frac{u-x}{k}\right)du$ where a probability density function $g(x)$ is such that

$$\int_{-\infty}^{\infty}g(x)dx=1, \text{otherwise } g(x)=0., \quad (3)$$

and obtained the proof of Weierstrass theorem [9], on extending the one variable function f to a bounded uniformly continuous function on $\mathbb{R}$, by letting $\frac{f(x)-f(a)}{x-a}=f(a)$ on an open interval $(a-1,a)$, $\frac{f(x)-f(b)}{x-b}=-f(b)$ over an open interval $(b,b+1)$, and $f(x)=0$ for all $x\in\mathbb{R}\setminus[a-1,b+1]$. Then for further analysis they set the density function as

$$g(x)=\lim_{n\to\infty}\frac{1}{C}\sum_{m=0}^{n}(-n)_m A_{n,m}x^{2m}, \quad (4)$$

where $A_{n,m}(\forall n\geq 0,\forall m\geq 0)$, is a bounded sequence,

$$(-n)_m=\frac{(-1)^m n!}{(n-m)!}, 0\leq m\leq n, \quad (5)$$

and for $m>n$, there is $(-n)_m=0$, the constant C may be determined by Eqn. (3).

On appealing Eqn. (4), they [5] also obtained a generating function that we now extend in the two variable operators

$$F(x,y)=\frac{1}{Ck_0}\sum_{n=0}^{\infty}\int_{-R}^{R}f(u)y^n\sum_{m=0}^{n}(-n)_m A_{n,m}\left(\frac{u-x}{k_0}\right)^{2m}du, k_0>0, x\in\mathbb{R}, |y|<1. \quad (6)$$

In the set of polynomials given in Eqn. (5), on setting $A_{n,m} = \frac{(\alpha+(\beta+1)n)!}{n!(\alpha+\beta n+m)!} B_m$, following generating functions fo $|y|<1, f \in [a,b]$, $\forall x,a,b \in \mathbb{R}$ has been obtained which is equivalent to the binomial distribution approximation operator $\forall (x,y) \in [a,b]\times[-1,1], a,b \in \mathbb{R}$

$$F(x,y) = \sum_{n=0}^{\infty} P_n(x) y^n = \frac{1}{Ck_0}\int_{-R}^{R} f(u) \sum_{n=0}^{\infty} y^n \sum_{m=0}^{n} (-1)^m \binom{\alpha+(\beta+1)n}{n-m} B_m \left(\frac{u-x}{k_0}\right)^{2m} du,$$

$$\forall (x,y) \in [a,b]\times[-1,1], a,b \in \mathbb{R}. \tag{7}$$

Now in (7), making an appeal to the Theorem due to Brown [1] on setting $\zeta = y(1+\zeta)^{\beta+1}$ and $\zeta(0)=0$, at $x=R$, our result becomes Volterra type integral operator for the kernel $K(x,u) = \sum_{n=0}^{\infty} B_n (-\zeta)^n \left(\frac{x-u}{k_0}\right)^{2n}$ in the form

$$F(x,\zeta) = \frac{(1+\zeta)^{\alpha+1}}{Ck_0(1-\beta\zeta)} \int_{-x}^{x} K(x,u) f(u)\, du, x \in \mathbb{R}, |y|<1. \tag{8}$$

Motivated by the above work, in this paper, we present a generalization of Lupas type operators for two variable functions and prove the Weierstrass approximation theorem for two variable functions $F(x,y)$ defined on a rectangle $[a,b]\times[c,d], \forall (x,y) \in \mathbb{R}^2, a,b \in \mathbb{R}$ and then finally analyze some of its properties, relations and applications in finding out the generating functions and Lupas- Durrmeyer type operators in two variables.

2. Weierstrass Approximations for Two Variable Functions.

In this section, in order to describe Weierstrass approximations for two variable functions, we define a transformation formula for a two variable function $F(x,y), \forall (x,y) \in \mathbb{R}^2$ in the form

$$H\{F(u,v); k_1,k_2,x,y\} = \frac{1}{k_1 k_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) G\left(\frac{u-x}{k_1}, \frac{v-y}{k_2}\right) du dv, \text{ where}$$

$$k_1, k_2 > 0. \tag{9}$$

A probability density function $G(x,y)$ is defined by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y)\, dx dy = 1, \text{ otherwise } G(x,y)=0. \tag{10}$$

First, we state and prove the following theorem for any two variable continuous function $F(x,y), \forall (x,y) \in \mathbb{R}^2$:

Theorem 1. $F(x,y)$ such that $F(x,y):[a,b]\times[c,d] \to \mathbb{R}^2; a,b,c,d \in \mathbb{R}$; $\forall (x,y) \in \mathbb{R}^2$, then $F(x,y)$ be a uniform limit of matrix of polynomials on a rectangle $[a,b]\times[c,d]$.

Proof. To prove this theorem we perform two steps and then using results of Step 1 and Step 2, we approach to the proof of this Theorem:

Step 1. Since with aid of the density function $G(x,y)$ given in Eqn. (10), we may put

$$F(x,y) = \frac{1}{k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y) G\left(\frac{u-x}{k_1}, \frac{v-y}{k_2}\right) dudv, \text{ where } k_1, k_2 > 0. \tag{11}$$

Therefore, for any $k_0 \in Max\{k_1, k_2\}$, letting $|u-x| < \delta_1$ and $|v-y| < \delta_2$, and taking $\delta_0 \in Min\{\delta_1, \delta_2\}$ such that $|F(u,v) - F(x,y)| \leq \epsilon_0 / 4$, $|H\{F(u,v);k_1,k_2,x,y\} - H\{F(u,v);k_0,k_0,x,y\}| < \epsilon_0 / 4$ and $|F(x,y)| \leq M/2$, $\forall (x,y) \in \mathbb{R}^2$, we define the operator

$$H\{F(u,v);k_0,k_0,x,y\} = \frac{1}{(k_0)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) G\left(\frac{u-x}{k_0}, \frac{v-y}{k_0}\right) dudv. \tag{12}$$

Then, with the aid of Eqns. (9), (11) and (12), we find that

$$|H\{F(u,v);k_1,k_2,x,y\} - F(x,y)|$$

$$< \frac{\epsilon_0}{4} + \frac{1}{(k_0)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v) - F(x,y)| G\left(\frac{u-x}{k_0}, \frac{v-y}{k_0}\right) dudv$$

$$= \frac{\epsilon_0}{4} + \frac{1}{(k_0)^2}\int_{|u-x|\leq\delta_0}\int_{|u-y|\leq\delta_0} |F(u,v) - F(x,y)| G\left(\frac{u-x}{k_0}, \frac{v-y}{k_0}\right) dudv$$

$$+ \frac{1}{(k_0)^2}\int_{|u-x|\leq\delta_0}\int_{|u-y|\leq\delta_0} |F(u,v) - F(x,y)| G\left(\frac{u-x}{k_0}, \frac{v-y}{k_0}\right) dudv$$

$$= \frac{\epsilon_0}{4} + \frac{\epsilon_0}{4} + \frac{M}{(k_0)^2}\int_{|u-x|\leq\delta_0}\int_{|u-y|\leq\delta_0} G\left(\frac{u-x}{k_0}, \frac{v-y}{k_0}\right) dudv \tag{13}$$

Now in above Eqn. (13), let $(u-x)/k_0=\xi$ and $(v-y)/k_0=\eta$, i.e. $|u-x|=k_0|\xi|$ and $|v-y|=k_0\eta$ and then due to Eqn. (12), we have $k_0/\delta_0|\xi|\geq 1$ and $k_0/\delta_0|\eta|\geq 1$, and therefore we obtain

$$\left|H\left\{F(u,v);k_1,k_2,x,y\right\}-F(x,y)\right|<\frac{\epsilon_0}{2}+\frac{M(k_0)^2}{(\delta_0)^2}\int_{|\xi|\geq\frac{\delta_0}{k_0}}\int_{|\xi|\geq\frac{\delta_0}{k_0}}|\xi||\eta|G(\xi,\eta)\,d\xi d\eta . \quad (14)$$

Now, in relation. (14) on defining $|\xi|=\begin{cases}-\xi,\xi<0,\\ \xi,\xi<0;\end{cases}$ and $|\eta|=\begin{cases}-\eta,\eta<0,\\ \eta,\eta<0;\end{cases}$, we derive

$$\left|H\left\{F(u,v);k_1,k_2,x,y\right\}-F(x,y)\right|<$$

$$\frac{\epsilon_0}{2}+\frac{M(k_0)^2}{(\delta_0)^2}\int_{\xi=\frac{\delta_0}{k_0}}^{\infty}\int_{\eta=\frac{\delta_0}{k_0}}^{\infty}\xi\eta G(\xi,\eta)\,d\xi d\eta+\frac{M(k_0)^2}{(\delta_0)^2}\int_{\xi=\frac{\delta_0}{k_0}}^{\infty}\int_{\eta=\frac{\delta_0}{k_0}}^{\infty}\xi\eta G(-\xi,-\eta)\,d\xi d\eta . \quad (15)$$

Again if $G(-\xi,-\eta)=\begin{cases}-G(\xi,\eta), G \text{ is odd,}\\ G(\xi,\eta), G \text{ is even;}\end{cases}$ and $\int_{\xi=\frac{\delta_0}{k_0}}^{\infty}\int_{\eta=\frac{\delta_0}{k_0}}^{\infty}\xi\eta G(\xi,\eta)\,d\xi d\eta$ $=L(\delta_0/k_0,\delta_0/k_0)<\infty$, from Eqn. (15), we establish

$$\left|H\left\{F(u,v);k_1,k_2,x,y\right\}-F(x,y)\right|<\begin{cases}\dfrac{\epsilon_0}{2} & ,G \text{ is odd,}\\[2ex] \dfrac{\epsilon_0}{2}+\dfrac{2M(k_0)^2}{(\delta_0)^2}L\left(\dfrac{\delta_0}{k_0},\dfrac{\delta_0}{k_0}\right) & ,G \text{ is even.}\end{cases} \quad (16)$$

Now letting $k_0=\dfrac{\delta_0}{2}\sqrt{\dfrac{\epsilon_0}{ML(\delta_0/k_0,\delta_0/k_0)}}$ in Eqn. (16), we get

$$\left|H\left\{F(u,v);k_1,k_2,x,y\right\}-F(x,y)\right|<\begin{cases}\dfrac{\epsilon_0}{2}, & G \text{ is odd,}\\ \epsilon_0, & G \text{ is even.}\end{cases} \quad (17)$$

Therefore, for an arbitrary $\epsilon_0\to 0$, we obtain

$$H\left\{F(u,v);k_1,k_2,x,y\right\}\to F(x,y). \quad (18)$$

Step 2. Here, we extend the function $F(x,y)$ to a bounded uniformly continuous function on $\mathbb{R}^2,\forall x,y\in\mathbb{R}$. This is accomplished here with the aid of following extended function.

Let $\dfrac{F(x,y)-F(a,y)}{x-a}=F(a,y), y$ is fixed on interval $(a-1,a)$ and

$\dfrac{F(x,y)-F(x,b)}{x-b}=-F(x,b), x$ is fixed on open interval $(b,b+1)$, again,

$\dfrac{F(x,y)-F(c,y)}{x-c}=F(c,y)$, for fixed y on open interval $(c-1,c)$ and

$\dfrac{F(x,y)-F(x,d)}{x-d}=-F(x,d), x$ is fixed on open interval $(d,d+1)$ and

$F(x,y)=0$ for all $(x,y)\in\mathbb{R}^2\setminus[a-1,b+1]\times[c-1,d+1]$.

Then, particularly, let $F(u,v)=0$ for $|u|>R, |v|>R, R>0$, so that from Eqn. (12) we may write

$$H\{F(u,v);k_0,k_0,x,y\}=\frac{1}{(k_0)^2}\int_{-R}^{R}\int_{-R}^{R}F(u,v)G\left(\frac{u-x}{k_0},\frac{v-y}{k_0}\right)dudv. \tag{19}$$

Again, let $G(x,y)=\lim_{n_1,n_2\to\infty}\dfrac{1}{C}\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}(-n_1)_{m_1}(-n_2)_{m_2}A[m_1,n_1;m_2,n_2]\,x^{2m_1}y^{2m_2}$,

where $A[m_1,n_1;m_2,n_2](n_i\geq 0, m_i\geq 0, \forall i=1,2)$ is a bounded matrix and

$(-n)_m=\dfrac{(-1)^m n!}{(n-m)!}, 0\leq m\leq n$, and for $m>n$, there is $(-n)_m=0$, and C is any constant which may be found by definition of probability density function $G(x,y)$ given in Eqn. (10). Also, for all $|x|\leq R_1\leq R$, and all $|u|\leq R_1\leq R$, we have $|u-x|\leq 2R$, and for all $|y|\leq R_2\leq R$, and all $|v|\leq R_2\leq R$, we have $|v-y|\leq 2R$, so that, $|(v-y)/k_0|\leq 2R/k_0, R=\min\{R_1,R_2\}$, hence then on the square $[-2R/k_0,2R/k_0]\times[-2R/k_0,2R/k_0]$, for an arbitrary ϵ_0, we get

$$\left|\frac{1}{(k_0)^2}G\left(\frac{u-x}{k_0},\frac{v-y}{k_0}\right)-\frac{1}{C(k_0)^2}\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}(-n_1)_{m_1}(-n_2)_{m_2}A[m_1,n_1;m_2,n_2]\left(\frac{u-x}{k_0}\right)^{2m_1}\left(\frac{v-y}{k_0}\right)^{2m_2}\right| < \frac{\epsilon_0}{8R^2M}. \tag{20}$$

Thus with the help of Eqns. (19) and (20), we have

$$\left|H\{F(u,v);k_0,k_0,x,y\}-\frac{1}{c(k_0)^2}\int_{-R}^{R}\int_{-R}^{R}F(u,v)\right.$$

$$\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}(-n_1)_{m_1}(-n_2)_{m_2}\,A[m_1,n_1;m_2,n_2]\left(\frac{u-x}{k_0}\right)^{2m_1}\left(\frac{v-y}{k_0}\right)^{2m_2}\Bigg| < \frac{\in_0}{4}. \tag{21}$$

Now, in above Eqn. (21), we define a matrix of polynomials in the form

$$P_{n_1,k_0,n_2,k_0}(x,y)=\frac{1}{c(k_0)^2}\int_{-R}^{R}\int_{-R}^{R}F(u,v)$$

$$\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}(-n_1)_{m_1}(-n_2)_{m_2}\,A[m_1,n_1;m_2,n_2]\left(\frac{u-x}{k_0}\right)^{2m_1}\left(\frac{v-y}{k_0}\right)^{2m_2}dudv$$

and get

$$\left|H\{F(u,v);k_0,k_0,x,y\}-P_{n_1,k_0,n_2,k_0}(x,y)\right| < \frac{\in_0}{4}. \tag{22}$$

Therefore,

$$\left|H\{F(u,v);k_1,k_2,x,y\}-P_{n_1,k_0,n_2,k_0}(x,y) < \left|H\{F(u,v);k_1,k_2,x,y\}-H\{H(u,v);k_0,k_0,x,y\}\right|\right|$$

$$+\left|H\{F(u,v);k_0,k_0,x,y\}-P_{n_1,k_0,n_2,k_0}(x,y)\right| < \frac{\in_0}{2}. \tag{23}$$

Finally, using Eqns. (22), (23), we find the result

$$\left|F(x,y)-P_{n_1,k_0,n_2,k_0}(x,y)\right| < \left|F(x,y)-H\{F(u,v);k_1,k_2,x,y\}\right|+$$

$$\left|H\{F(u,v);k_1,k_2,x,y\}-P_{n_1,k_0,n_2,k_0}(x,y)\right| = \frac{\in_0}{2}+\frac{\in_0}{2}=\in_0. \tag{24}$$

Or, in other words from Eqn. (24), for an arbitrary $\in_0 \to 0$, we find

$$F(x,y)\to P_{n_1,k_0,n_2,k_0}(x,y),\forall(x,y)\in\mathbb{R}^2. \tag{25}$$

Remark 1. Further, let $A[m_1,n_1;m_2,n_2]=A[m_1,n_1;0,0]=A_{m_1,n_1}, k_2=1,$ again, let $F(x,y)=g(y), f(x)g(y)$ is defined in Eqn. (3), and then making an application of the Eqns. (9),(10),(20), and (22), we obtain the following equivalent results due to Kumar and Pathan [5]

$$P_{n_1,k_1,0,1}(x,y)=\frac{1}{Ck_1}\int_{-R}^{R}f(u)\sum_{m_1=0}^{n_1}(-n_1)_{m_1}A_{m_1,n_1}\left(\frac{u-x}{k_1}\right)^{2m_1}du \text{ and}$$

$$H\{f(u)g(v);k_1,1,x,y\}=$$

$$H_{k_1}f(x)=\frac{1}{Ck_1}\lim_{n_1\to\infty}\int_{-R}^{R}f(u)\sum_{m_1=0}^{n_1}(-n_1)_{m_1}A_{m_1,n_1}\left(\frac{u-x}{k_1}\right)^{2m_1}du. \tag{26}$$

Remark 2. Let $A[m_1,n_1;m_2,n_2] = \frac{(n_1-m_1)!}{n_1!m_1!}\frac{(n_2-m_2)!}{n_2!m_2!}$, then by definition of the probability density function $G(x,y)$ given in Eqns. (9), (10), (20), and (22), we find $C=\pi$ and in particular, we have

$$P_{n_1,k_1,n_2,k_2}(x,y) = \frac{1}{\pi k_1 k_2}\int_{-R}^{R}\int_{-R}^{R} F(u,v)\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}\frac{(-1)^{m_1}}{m_1!}\frac{(-1)^{m_2}}{m_2!}\left(\frac{u-x}{k_1}\right)^{2m_1}\left(\frac{v-y}{k_2}\right)^{2m_2} du dv$$

and $$H\{F(u,v);k_1,k_2,x,y\} = \frac{1}{\pi k_1 k_2}\int_{-R}^{R}\int_{-R}^{R} F(u,v)e^{-\left(\frac{u-x}{k_1}\right)^2} e^{-\left(\frac{v-y}{k_2}\right)^2} du dv \tag{27}$$

Again, as $R\to\infty$, these results of Eqn. (26) may be written by **Weierstrass approximations for two variable functions**

$$P_{n_1,k_1;n_2,k_2}(x,y) = \frac{1}{\pi k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}\frac{(-1)^{m_1}}{m_1!}\frac{(-1)^{m_2}}{m_2!}\left(\frac{u-x}{k_1}\right)^{2m_1}\left(\frac{v-y}{k_2}\right)^{2m_2} du dv$$

and

$$H\{F(u,v);k_1,k_2,x,y\} = \frac{1}{\pi k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{-\left(\frac{u-x}{k_1}\right)^2} e^{-\left(\frac{v-y}{k_2}\right)^2} du dv. \tag{28}$$

Remark 3. Further, let $A[m_1,n_1;m_2,n_2] = A[m_1,n_1;0,0] = \frac{(n_1-m_1)!}{n_1!m_1!}, k_2=1,$ then $C=\sqrt{\pi}$ and then from Eqn. (28), we have

$$P_{n_1,k_1;0,1}(x,y) = \frac{1}{\sqrt{\pi}k_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\sum_{m_1=0}^{n_1}\frac{(-1)^{m_1}}{m_1!}\left(\frac{u-x}{k_1}\right)^{2m_1} du dv \text{ and}$$

$$H\{F(u,v);k_1,1,x,y\} = \frac{1}{\sqrt{\pi}k_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{-\left(\frac{u-x}{k_1}\right)^2} du dv \tag{29}$$

Remark 4. Further, let $A[m_1,n_1;m_2,n_2] = A[m_1,n_1;0,0] = \frac{(n_1-m_1)!}{n_1!m_1!}, k_2=1,$ then $C=\sqrt{\pi}$ and again let $F(x,y)=f(x)g(y), g(y)$ is defined by Eqn (3), then on making an appeal to Eqns. (28) and (29) we obtain the results equivalent to Schep [8] given in Eqn. (5)

$$P_{n_1,k_1,0,1}(x,y) = \frac{1}{\sqrt{\pi}k_1}\int_{-\infty}^{\infty} f(u)\sum_{m_1=0}^{n_1}\frac{(-1)^{m_1}}{m_1!}\left(\frac{u-x}{k_1}\right)^{2m_1} du$$

and

$$H\{f(u)g(v);k_1,1,x,y\}=\frac{1}{\sqrt{\pi}k_1}\int_{-\infty}^{\infty}f(u)e^{-\left(\frac{u-x}{k_1}\right)^2}du. \tag{30}$$

3. Properties and Relations of Weierstrass Type Approximation of Two Variable Functions. Consider the transformation formula given by Eqn. (9)

$$H\{F(u,v);k_1,k_2,x,y\}=\frac{1}{k_1k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)G\left(\frac{u-x}{k_1},\frac{v-y}{k_2}\right)dudv,$$

where $k_1,k_2>0$ and set $F(x,y)=1$.

Then we find that $H\{1;k_1,k_2,x,y\}=1.$ (31)

Theorem 2. Let $G\left(\frac{u-x}{k_1},\frac{v-y}{k_2}\right)$ be such that

$$\mu_{r_1,r_2}=\frac{1}{k_1k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u-x}{k_1}\right)^{r_1}\left(\frac{v-y}{k_2}\right)^{r_2}G\left(\frac{u-x}{k_1},\frac{v-y}{k_2}\right)dudv<\infty,\forall r_1,r_2\in\mathbb{R},$$

then $H\{F(u,v);k_1,k_2,x,y\}$

$$=\sum_{N=0}^{\infty}\left\{\frac{1}{N!}\sum_{M=0}^{N}\binom{N}{M}(k_1)^{N-M}(k_2)^{M}\mu_{N-M,M}\frac{\partial^N F}{\partial u^{N-M}\partial v^{M}}(x,y)\right\},\ \forall M\le N.$$

and $H\{F(u,v);k_1,k_2,x,y\}=0,\forall M>N.$ (32)

Proof. The general expression for the Taylor series in two variables in the neighbourhood of (x,y) is given by (see Pipe [6])

$$F(u,v)=\sum_{N=0}^{\infty}\left\{\frac{1}{N!}\sum_{M=0}^{N}\binom{N}{M}(u-x)^{N-M}(v-y)^{M}\frac{\partial^N F(u,v)}{\partial u^{N-M}\partial v^{M}}\Big|at\,(u,v)=(x,y)\right\}$$

$$=\sum_{N=0}^{\infty}\left\{\frac{1}{N!}\sum_{M=0}^{N}\binom{N}{M}(u-x)^{N-M}(v-y)^{M}\frac{\partial^N F}{\partial u^{N-M}\partial v^{M}}(x,v)\right\}.$$

If we multiply both sides by $\frac{1}{k_1k_2}G\left(\frac{u-x}{k_1},\frac{v-y}{k_2}\right)$ and then integrate two times with respect to u from $u=-\infty$ to $u=\infty$, again same with respect to v

from $\upsilon = -\infty$ to $\upsilon = \infty$, and then use Eqn. (9), Theorem 2 and the definition

$$\binom{N}{M} = \begin{cases} \dfrac{N!}{M!(N-M)!}, M \le N \\ 0, M > N \end{cases}$$, we obtain the formula (32).

Theorem 3. Let a number

$$\mu_{r_1,r_2} = \frac{1}{k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u-x}{k_1}\right)^{r_1}\left(\frac{\upsilon-y}{k_2}\right)^{r_2} G\left(\frac{u-x}{k_1},\frac{\upsilon-y}{k_2}\right)du d\upsilon$$

is real and finite, then in the neighbourhood of point (x,y), the continuity of $F(x,y)$ with $H\{F(u,\upsilon);k_1,k_2,x,y\}$ gives

$$\left|H\{F(u,\upsilon);k_1,k_2,x,y\} - F(x,y)\right|$$
$$= \sum_{N=1}^{\infty}\left\{\frac{1}{N!}\sum_{M=1}^{N}\binom{N}{M}(k_1)^{N-M}(k_2)^{M}\mu_{N-M,M}\frac{\partial^N F}{\partial u^{N-M}\partial \upsilon^M}(x,y)\right\} \quad \forall M \le N. \qquad (33)$$

Proof. Making an appeal to (31) and (32), we can easily prove the result (33).

Theorem 4. Let a number

$$\mu_{r_1,r_2} = \frac{1}{k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u-x}{k_1}\right)^{r_1}\left(\frac{\upsilon-y}{k_2}\right)^{r_2} G\left(\frac{u-x}{k_1},\frac{\upsilon-y}{k_2}\right)du d\upsilon$$

is real and finite, and then in the neighbourhood of point (x,y), the expected value of any two variable function $F(x,y)$ may be found by the formula

$$\frac{1}{k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,\upsilon)G\left(\frac{u-x}{k_1},\frac{\upsilon-y}{k_2}\right)du d\upsilon$$
$$= \sum_{N=0}^{\infty}\left\{\frac{1}{N!}\sum_{M=0}^{N}\binom{N}{M}(k_1)^{N-M}(k_2)^{M}\mu_{N-M,M}\frac{\partial^N F}{\partial u^{N-M}\partial \upsilon^M}(x,y)\right\}\forall M \le N.$$

(34)

Proof. Making an appeal to (31) and (32), we prove the result (34).

Theorem 5. Let $F(x,y) = F_1(x)F_2(y)$ and $G(x,y) = G_1(x)G_2(y)$, and a number

$$\mu^{*}_{r_1,r_2} = \frac{1}{k_1 k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u-x}{k_1}\right)^{r_1}\left(\frac{\upsilon-y}{k_2}\right)^{r_2} G_1\left(\frac{u-x}{k_1}\right)G_2\left(\frac{\upsilon-y}{k_2}\right)du d\upsilon$$

is real and finite, and then in the neighbourhood of point (x,y) the expected values of product of two functions may be found by the formula

$$\left(\frac{1}{k_1}\int_{-\infty}^{\infty} F_1(u)G_1\left(\frac{u-x}{k_1}\right)du\right)\left(\frac{1}{k_2}\int_{-\infty}^{\infty} F_2(v)G_2\left(\frac{v-y}{k_2}\right)dv\right)$$

$$=\sum_{N=0}^{\infty}\left\{\frac{1}{N!}\sum_{M=0}^{N}\binom{N}{M}(k_1)^{N-M}(k_2)^{M}\dot{\mu}_{N-M,M}\frac{\partial^{N-M}F_1}{\partial u^{N-M}}(x)\frac{\partial^{M}F_2}{\partial v^{M}}(y)\right\} \quad \forall M\le N. \qquad (35)$$

Proof. Making an appeal to the Theorem 4, and then choosing $F(x,y)=F_1(x)F_2(y)$ and $G(x,y)=G_1(x)G_2(y)$, we prove the result (35).

Theorem 6. Let $F(x,y)=F_1(x)F_2(y)$ and a number

$$\mu_{r_1,r_2}=\frac{1}{k_1k_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u-x}{k_1}\right)^{r_1}\left(\frac{v-y}{k_2}\right)^{r_2}F_1\left(\frac{u-x}{k_1}\right)F_2\left(\frac{v-y}{k_2}\right)dudv$$

is real and finite, then in the neighbourhood of point (x,y), the continuity of $F(x,y)$ with $H\{F_1(u)F_2(v);k_1,k_2,x,y\}$ may be numerically computed by

$$\left|H\{F_1(u)F_2(v);k_1,k_2,x,y\}-F_1(x)F_2(y)\right|$$

$$=\sum_{N=1}^{\infty}\left\{\frac{1}{N!}\sum_{M=1}^{N}\binom{N}{M}(k_1)^{N-M}(k_2)^{M}\mu_{N-M,M}\phi_{N,M}(x,y)\right\},\forall M\le N,$$

where

$$\phi_{N,M}(x,y)=\lim_{h_1\to 0}\left(\frac{1}{h_1}\right)^{N-M}\sum_{i_1=0}^{\infty}(-1)^{i_1}\binom{n-m}{i_1}F_1(x-i_1h_1)$$

$$\lim_{h_2\to 0}\left(\frac{1}{h_2}\right)^{N-M}\sum_{i_2=0}^{\infty}(-1)^{i_2}\binom{m}{i_2}F_2(y-i_2h_2). \qquad (36)$$

Proof. In Theorem 3, taking $F(x,y)=F_1(x)F_2(y)$ and then making an appeal to (32), we get

$$\left|H\{F_1(u)F_2(v);k_1,k_2,x,y\}-F_1(x)F_2(y)\right|$$

$$=\sum_{N=1}^{\infty}\left\{\frac{1}{N!}\sum_{M=1}^{N}\binom{N}{M}(k_1)^{N-M}(k_2)^{M}\mu_{N-M,M}\frac{\partial^{N-M}F_1}{\partial u^{N-M}}(x)\frac{\partial^{M}F_2}{\partial v^{M}}(y)\right\}. \qquad (37)$$

Now, making an appeal to following theorem due to Diethem [2, Theorem 2. D., p. 42]:

Let $n \in \mathbb{N}$ (The set of Natural numbers), the n^{th} differential coefficient of the function f is exists and $a < x < b, a, b \in \mathbb{R}$, then

$$D^n f(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} f(x - kh) \tag{38}$$

and the result (37), we prove the result (36).

4. Applications.

1. Generating Functions. Making an appeal to the matrix polynomials defined in (22) such that

$$P_{n_1,k_1;n_2,k_2}(x,y) = \frac{1}{Ck_1k_2} \int_{-R}^{R} \int_{-R}^{R} F(u,v)$$

$$\sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} (-n_1)_{m_1} (-n_2)_{m_2} A[m_1, n_1; m_2, n_2] \left(\frac{u-x}{k_1}\right)^{2m_1} \left(\frac{v-y}{k_2}\right)^{2m_2} dudv \tag{39}$$

and using the technique due to Kumar and Pathan [5]. (See Eqns. (6), (7) and (8)), we may find several generating functions $\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} t_1^{n_1} t_2^{n_2} P_{n_1,k_1;n_2,k_2}(x,y), |t_1| < 1, |t_2| < 1$, for hypergeometric functions of two variables,

2. Two Variable Lupas- Durrmeyer type Operators. In the Eqn. (39), for large R, if we replace $\frac{1}{C} = \binom{n_1 + m_1 - 1}{m_1} \frac{x^{m_1}}{(1+x)^{m_1+n_1}} \binom{n_2 + m_2 - 1}{m_2} \frac{y^{m_2}}{(1+x)^{m_2+n_2}}$,

$$\forall x \geq 0, y \geq 0, A[m_1, n_1; m_2, n_2] = \frac{(n_1 - 1)}{(-n_1)_{m_1}} \binom{n_1 + m_1 - 1}{m_1} \frac{(n_2 - 1)}{(-n_2)_{m_2}} \binom{n_2 + m_2 - 1}{m_2}$$

bounded for $\forall (n_i \geq 0, m_i \geq 0, i = 1, 2)$, and put $\frac{((u-x)/k_1)^{1/2}}{(1 + (u-x)/k_1)^{1/2 + n_1/2m_1}}$ in place of $\frac{u-x}{k_1}$ and $\frac{((v-y)/k_2)^{1/2}}{(1 + (v-y)/k_2)^{1/2 + n_2/2m_2}}$ in place of $\frac{v-y}{k_2}$ $\forall u \geq 0, v \geq 0$ and $\forall (n_i \geq 0, m_i \geq 0, i = 1, 2)$, we get $P_{n_1,k_1;n_2,k_2}(x,y) = \frac{1}{k_1k_2}(n_1 - 1)(n_2 - 1) \int_0^{\infty} \int_0^{\infty} F(u,v)$

$$\sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} C_{m_1,m_2}^{n_1,n_2}(x,y) C_{m_1,m_2}^{n_1,n_2}\left(\frac{u-x}{k_1}, \frac{v-y}{k_2}\right) dudv.$$

Here, we have

$$C_{m_1,m_2}^{n_1,n_2}(x,y) = \binom{n_1+m_1-1}{m_1}\binom{n_2+m_2-1}{m_2}\frac{x^{m_1}}{(1+x)^{m_1+n_1}}\frac{y^{m_2}}{(1+y)^{m_2+n_2}}. \tag{40}$$

now, making an appeal to (20), (21) and (40), we get two variable Lupas-Durrmeyer type operators

$$H\{F(u,v);k_1,k_2,x,y\} = \lim_{n_1,n_2\to\infty}(n_1-1)(n_2-1)\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}C_{m_1,m_2}^{n_1,n_2}(x,y)$$

$$\frac{1}{k_1k_2}\int_0^\infty\int_0^\infty F(u,v)C_{m_1,m_2}^{n_1,n_2}\left(\frac{u-x}{k_1},\frac{v-y}{k_2}\right)dudv\ . \tag{41}$$

(For one variable Lupas- Durrmeyer type operators see Gupta [3] and Gupta and Rassias [4]).

Again with the help of Theorem 2 and relations (40) and (41), we obtain the relation

$$H\{F(u,v);k_1,k_2,x,y\} = \lim_{n_1,n_2\to\infty}\sum_{N=0}^{\infty}\sum_{M=0}^{N}(k_1)^{N-M}(k_2)^M\frac{(n_1-N+M-2)!(n_2-M-2)!}{(n_1-2)!(n_2-2)!(1+x)^{n_1}(1+y)^{n_2}}$$

$${}_2F_1\left[n_1,1+N-M;1;\frac{x}{1+x}\right]{}_2F_1\left[n_2,1+M;1;\frac{y}{1+y}\right]\frac{\partial^N F}{\partial u^{N-M}\partial v^M}(x,y)\,\forall M\le N. \tag{42}$$

Conclusions. In this paper, we presented a generalization of Lupas type operators for two variable functions and proved the Weierstrass approximation theorem for two variable functions motivated by the work of Kumar and Pathan[5]. Using these results, an attempt can be made to obtain further generalizations and studies for other type of operators found in the literature, Volterra integral equations and their solutions. Also, we may obtain approximate values of several two variable functions and their generating functions.

ACKNOWLEDGEMENT

The second author M.A.Pathan would like to thank the Department of Science and Technology, Government of India, for the financial assistance for this work under project number SR/S4/MS:794/12 and the Centre for Mathematical Sciences for the facilities.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

A MULTIVARIABLE ANALOGUE OF A CLASS OF POLYNOMIALS

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(Received : December 16, 2014)

ABSTRACT

In the present paper, we introduce and study a multivariable analogue of polynomials due to Chandel and Chandel (*Rev. Tec. Ing., Univ. Zulia,* **7(1)** (1984), 63-67) and discuss its interesting special cases.

2010 Mathematics Subject Classification : 33C50

Keywords : Multivariable analogue, Panda's polynomials, Agrawal polynomials, Polynomials of Chandel and Chandel.

1. Introduction. Agrawal [1] introduced the polynomials defined by

$$(1.1)\quad \left(1-pt^{q}\right)^{-c}\exp\left[-\frac{r^{r}xt}{\left(1-pt^{q}\right)^{r}}\right]=\sum_{n=0}^{\infty}f_{n}^{c}\left(x;p,q,r\right)\,t^{n}.$$

Also Panda [4] introduced the class of polynomials defined by

$$(1.2)\quad \left(1-t\right)^{-c}G\left[\frac{xt^{s}}{\left(1-t\right)^{r}}\right]=\sum_{n=0}^{\infty}g_{n}^{c}\left(x,r,s\right)\,t^{n},$$

where c is any arbitrary parameter, r is any integer positive or negative and r=1,2,3,....

Further, Chandel and Chandel [2] introduced the polynomials defined by

$$(1.3)\quad \left(1-pt^{q}\right)^{-c}G\left[\frac{xt}{\left(1-pt^{q}\right)^{r}}\right]=\sum_{n=0}^{\infty}g_{n}^{c}\left(x,p,q,r\right)t^{n},$$

where

(1.4) $G(z)=\sum_{n=0}^{\infty}\gamma_n z^n,(\gamma_0 \neq 0),$

q is any positive integer and other parameters are unrestricted in general.

The definition (1.3) and (1.4) were motivated by earlier work (1.1) due to Agrawal [1], who considers the special case of (1.3) where $\gamma_n=\frac{(-1)^n r^{rn}}{n!}$ and also the work on (1.2) due to Panda [4] who considers only the case when $s=1,2,3,...$ and $p=1$, while in (1.3) Chandel and Chandel [2] discuss the case when $s=1/q;q=1,2,3,...$

Chandel and Sahgal [3] introduced a multivarialbe analogue of Panda's polynomails ([2]; also see (1.2)) by generating function

$$(1.5)\quad (1-t_1)^{-c_1}...(1-t_m)^{-c_m}\left[1-\frac{x_1t_1^{s_1}}{(1-t_1)^{r_1}}-...-\frac{x_mt_m^{s_m}}{(1-t_m)^{r_m}}\right]^{-b}$$

$$=\sum_{n_1,...,n_m=0}^{\infty}\Gamma_{n_1,...,n_m}^{(b;c_1,...,c_m;r_1,...,r_m;s_1,...,s_m)}(x_1,...,x_m)t_1^{n_1}...t_m^{n_m},$$

where $b,c_1,...,c_m$ are any parameters $r_1,...,r_m$ are any integers positive or negative, while $s_1,...,s_m$ are positive integers.

They also discussed their generalization as

$$(1.6)\quad (1-t_1)^{-c_1}...(1-t_m)^{-c_m}G\left(\frac{x_1t_1^{s_1}}{(1-t_1)^{r_1}}+...+\frac{x_mt_m^{s_m}}{(1-t_m)^{r_m}}\right)$$

$$=\sum_{n_1,...,n_m=0}^{\infty}G_{n_1,...,n_m}^{(c_1,...,c_m;r_1,...,r_m;s_1,...,s_m)}(x_1,...,x_m)t_1^{n_1}...t_m^{n_m},$$

where $c_1,...,c_m$ are any parameters, $r_1,...,r_m$ are any integers, positive or negative, while $s_1,...,s_m$ are any positive integers and $x_1,...,x_m$ are any variables real or complex and $G(z)$ is given by (1.4),

Here in the present paper motivated by earlier works (1.3) and (1.6), we introduce a multivariable analogue of the class of polynomials due to Chandel and Chandel [2] defined by (1.3).

Our multivariable polynomials are defined through generating relation

$$(1.7)\quad \left(1-p_1t_1^{q_1}\right)^{-c_1}\dots\left(1-p_mt_m^{q_m}\right)^{-c_m} G\left(\frac{x_1t_1}{\left(1-p_1t_1^{q_1}\right)^{r_1}}+\dots+\frac{x_mt_m}{\left(1-p_mt_m^{q_m}\right)^{r_m}}\right)$$

$$=\sum_{n_1,\dots,n_m=0}^{\infty} g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)t_1^{n_1}\dots t_m^{n_m}\ ,$$

where $G(z)$ is given by (1.4) and $q_1,\dots,q_m$ are positive integers while other parameters are unrestricted in general.

It is remarkable that in (1.6) authors consider only the case when $s_i=1,2,3,\dots$ and $p_i=1$ $(i=1,\dots,m)$ while in (1.7), we consider the case when $s_i=1/q_i$, $q_i=1,2,3,\dots$ $(i=1,2,\dots,\mathrm{m})$.

2. Explicit Form. By an appeal to generating relation (1.7), we have

$$\sum_{n_1,\dots,n_m=0}^{\infty} g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)t_1^{n_1}\dots t_m^{n_m}$$

$$=\sum_{N=0}^{\infty}\gamma_N\left[\frac{x_1t_1}{\left(1-p_1t_1^{q_1}\right)^{r_1}}+\dots+\frac{x_mt_m}{\left(1-p_mt_m^{q_m}\right)^{r_m}}\right]^N\left(1-p_1t_1^{q_1}\right)^{-c_1}\dots\left(1-p_mt_m^{q_m}\right)^{-c_m}$$

$$=\sum_{n_1,\dots n_m=0}^{\infty}\gamma_{n_1+\dots+n_m}\left(x_1t_1\right)^{n_1}\left(1-p_tt_1^{q_1}\right)^{-(c_1+r_1k_1)}\dots\left(x_mt_m\right)^{n_m}\left(1-p_mt_m^{q_m}\right)^{-(c_m+r_mk_m)}$$

$$=\sum_{n_1,\dots n_m=0}^{\infty}\sum_{k_1=0}^{[n_1/q_1]}\dots\sum_{k_m=0}^{[n_m/q_m]}\gamma_{n_1-k_1q_1,\dots,n_m-k_mq_m}\left(c_1+r_1\left(n_1-q_1k_1\right)\right)_{k_1}\dots\left(c_m+r_m\left(n_m-q_mk_m\right)\right)_{k_m}$$

$$x_1^{n_1-q_1k_1}\dots x_m^{n_m-q_mk_m}\frac{p_1^{k_1}}{k_1!}\dots\frac{p_m^{k_m}}{k_m!}t_1^{n_1}\dots t_m^{n_m}\ .$$

Thus equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$ both the sides, we get required explicit form

$$(2.1)\quad g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)$$

$$=\sum_{k_1=0}^{[n_1/q_1]}\dots\sum_{k_m=0}^{[n_m/q_m]}\gamma_{n_1-k_1q_1,\dots,n_m-k_mq_m}\left(c_1+r_1\left(n_1-q_1k_1\right)\right)_{k_1}\dots\left(c_m+r_m\left(n_m-q_mk_m\right)\right)_{k_m}$$

$$x_1^{n_1-q_1k_1}\dots x_m^{n_m-q_mk_m}\frac{p_1^{k_1}}{k_1!}\dots\frac{p_m^{k_m}}{k_m!}\ .$$

3. Other Applications of Generating Relations. Starting with the generating relation (1.7), we derive

$$(3.1)\quad g_{n_1}^{(c_1+c'_1,\dots,c_m+c'_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)$$

$$=\sum_{k_1=0}^{[n_1/q_1]}\cdots\sum_{k_m=0}^{[n_m/q_m]}\frac{\left(c_1\right)_{k_1}}{k_1!}\cdots\frac{\left(c_m\right)_{k_m}}{k_m!}g_{n_1-k_1q_1,\dots,n_m-k_mq_m}^{(c'_1,\dots,c'_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right).$$

Further an appeal to generating relation (1.7), shows that

$$(3.2)\quad g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)$$

$$=\sum_{k_1=0}^{[n_1/q_1]}\cdots\sum_{k_m=0}^{[n_m/q_m]}\frac{\left(c_1-b_1\right)_{k_1}}{k_1!}\cdots\frac{\left(c_m-b_m\right)_{k_m}}{k_m!}p_1^{k_1}\cdots p_m^{k_m}$$

$$g_{n_1-q_1k_1,\dots,n_m-q_mk_m}^{(b_1,\dots,b_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}\left(x_1,\dots,x_m\right).$$

4. Differential Recurrence Relations. Differentiating (1.7) partially with respect to x_i, we have

$$(4.1)\quad \left(1-p_1t_1^{q_1}\right)^{-c_1}\cdots\left(1-p_mt_m^{q_m}\right)^{-c_m}\frac{t_i}{\left(1-p_it_i^{q_i}\right)^{r_i}}G'$$

$$=\sum_{n_1,\dots,n_m=0}^{\infty}\frac{\partial}{\partial x_i}g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)t_1^{n_1}\cdots t_m^{n_m}.$$

Similarly, differentiating (1.7) partially with respect to x_j, we have

$$(4.2)\quad \left(1-p_1t_1^{q_1}\right)^{-c_1}\cdots\left(1-p_mt_m^{q_m}\right)^{-c_m}\frac{t_j}{\left(1-p_jt_j^{q_j}\right)^{r_j}}G'$$

$$=\sum_{n_1,\dots,n_m=0}^{\infty}\frac{\partial}{\partial x_j}g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)t_1^{n_1}\cdots t_m^{n_m}.$$

Thus eliminating G' from (4.1) and (4.2) and equating the coefficients of $t_1^{n_1}\cdots t_m^{n_m}$ both the sides, we establish

$$(4.3)\quad \sum_{k=0}^{\min[r_j,[n_j/q_j]]}\binom{r_j}{k}p_j^k\frac{\partial}{\partial x_j}g_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_{j-1},n_j-kq_j,n_{j+1},\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)$$

$$=\sum_{k=0}^{\min[r_i,[n_i/q_i]]}\binom{r_i}{k}p_i^k\frac{\partial}{\partial x_i}g_{n_1,\dots,n_{i-1},n_i-q_ik,n_{i+1},\dots,n_{j-1},n_j-1,n_{j+1},\dots,n_m}^{(c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m,r_1,\dots,r_m)}\left(x_1,\dots,x_m\right)$$

where $i\leq j$ and $i,j=1,\dots,m$.

By (4.1), we can further write

$$(4.4)\quad \sum_{n_1,\dots,n_m=0}^{\infty} \frac{\partial}{\partial x_i} g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)t_1^{n_1}\dots t_m^{n_m}$$

$$= \left(1-p_1t_1^{q_1}\right)^{-c_1}\dots\left(1-p_{i-1}t_{i-1}^{q_{i-1}}\right)^{-c_{i-1}}\left(1-p_it_i^{q_i}\right)^{-(c_i+r_i)}\left(1-p_{i+1}t_{i+1}^{q_{i+1}}\right)^{-c_{i+1}}\dots\left(1-p_mt_m^{q_m}\right)^{-c_m}$$

$$t_iG'$$

Also differenciating (1.7) partially with respect to t_i, we derive

$$(4.5)\quad c_ip_iq_i\sum_{n_1,\dots,n_m=0}^{\infty} g_{n_1,\dots,n_{i-1},n_i-q_i+1,n_{i+1},\dots,n_m}^{(c_1,\dots,c_{i-1},c_i+1,c_{i+1},\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)t_1^{n_1}\dots t_m^{n_m}$$

$$-\sum_{n_1,\dots,n_m=0}^{\infty}(n_i+1)\frac{\partial}{\partial x_i} g_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m)}(x_1,\dots,x_m)t_1^{n_1}\dots t_m^{n_m}$$

$$= -\left(1-p_1t_1^{q_1}\right)^{-c_1}\dots\left(1-p_it_i^{q_i}\right)^{-(c_i+r_i+1)}\dots\left(1-p_mt_m^{q_m}\right)^{-c_m}G'.$$

Eliminating G' from (4.4) and (4.5) and equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$ both the sides, we finally arrive at differential recurrence relation:

$$(4.6)\quad c_ip_iq_ig_{n_1,\dots,n_{i-1},n_i-q_i,n_{i+1},\dots,n_m}^{(c_1,\dots,c_{i-1},c_i+1,c_{i+1},\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$+\ (x_i-n_i)\frac{\partial}{\partial x_i} g_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$-p_i^2c_iq_ig_{n_1,\dots,n_{i-1},n_i-2q_i,n_{i+1},\dots,n_m}^{(c_1,\dots,c_{i-1},c_i+1,c_{i+1},\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$+p_i(n_i-q_i)\frac{\partial}{\partial x_i} g_{n_1,\dots,n_{i-1},n_i-q_i,n_{i+1},\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$+(r_iq_i-1)p_ix_i\frac{\partial}{\partial x_i} g_{n_1,\dots,n_{i-1},n_i-q_i,n_{i+1},\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=0, \qquad\qquad i=1,\dots,m.$$

5. Special Case I. For $\gamma_n = \dfrac{(-1)^n}{n!}$, (1.6) defines multivariable polynomials $E_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)$ as

$$(5.1)\quad \sum_{n_1,\dots,n_m=0}^{\infty} E_{n_1,\dots,n_m}^{(c_1,\dots,c_m;p_1\dots p_m;q_1,\dots,q_m,r_1,\dots,r_m)}(x_1,\dots,x_m)\,t_1^{n_1}\dots t_m^{n_m}$$

$$= \sum_{n_1,\dots,n_m=0}^{\infty} f_{n_1}^{c_1}\left(-x_1/r_1^{r_1};p_1,q_1,r_1\right)\dots f_{n_m}^{c_m}\left(-x_m/r_m^{r_m};p_m,q_m,r_m\right)t_1^{n_1}\dots t_m^{n_m}.$$

That is

(5.2) $E_{n_1,\ldots,n_m}^{(c_1,\ldots,c_m;p_1,\ldots,p_m;q_1,\ldots,q_m;r_1,\ldots,r_m)}(x_1,\ldots,x_m)$

$$= f_{n_1}^{c_1}\left(-x_1/r_1^{r_1};p_1,q_1,r_1\right)\ldots f_{n_m}^{c_m}\left(-x_m/r_m^{r_m};p_m,q_m,r_m\right),$$

where $f_n^c(x;p,q,r)$ are polynomials of Agrawal [1] defined by (1.1).

6. Special Case II. For $\gamma_n = \dfrac{(b)_n}{n!}$, (1.7) defines the multivariable polynomials $B_{n_1,\ldots,n_m}^{(b;c_1,\ldots,c_m;p_1,\ldots,p_m;q_1,\ldots,q_m;r_1,\ldots,r_m)}(x_1,\ldots,x_m)$ by generating relation

(6.1) $$\left(1-p_1t_1^{q_1}\right)^{-c_1}\ldots\left(1-p_mt_m^{q_m}\right)^{-c_m}\left[1-\frac{x_1t_1}{\left(1-p_1t_1^{q_1}\right)^{r_1}}-\ldots-\frac{x_mt_m}{\left(1-p_mt_m^{q_m}\right)^{r_m}}\right]^{-b}$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} B_{n_1,\ldots,n_m}^{(b;c_1,\ldots,c_m;p_1,\ldots,p_m;q_1,\ldots,q_m;r_1,\ldots,r_m)}(x_1,\ldots,x_m)\, t_1^{n_1}\ldots t_m^{n_m},$$

which suggests that

(6.2) $$\sum_{n_1=0}^{\infty} B_{n_1}^{(b;c_1;p_1;q_1;r_1)}(x_1)t_1^{n_1} = \left(1-p_1t_1^{q_1}\right)^{-c_1}\left[1-\frac{x_1t_1}{\left(1-p_1t_1^{q_1}\right)^{r_1}}\right]^{-b_1}.$$

Thus we have

(6.3) $B_{n_1}^{(b;c_1;p_1;q_1;r_1)}(x_1) = g_n^{(b,c_1)}(x_1,p_1,q_1,r_1)$,

where $g_n^{(b,c)}(x,p,q,r)$ are the polynomials due to Chandel and Chandel ([2],(3.1)), defined by generating relation

(6.4) $$\left(1-pt^q\right)^{-c}\left(1-\frac{xt}{1-pt^q}\right)^{-b} = \sum_{n=0}^{\infty} g_n^{(b,c)}(x,p,q,r)\, t^n$$

as special case for $\gamma_n = \dfrac{(b)_n}{n!}$ of the polynomials of Chandel and Chandel ([2],(1.3)) defined by (1.3).

Now by an appeal to generating relation (6.1), we derive

(6.5) $B_{n_1,\ldots,n_m}^{(b+b';c_1+c'_1,\ldots,c_m+c'_m;p_1,\ldots,p_m;q_1,\ldots,q_m;r_1,\ldots,r_m)}(x_1,\ldots,x_m)$

$$= \sum_{k_1=0}^{n_1}\cdots\sum_{k_m=0}^{n_m} B_{k_1,\ldots,k_m}^{(b;c_1,\ldots,c_m;p_1,\ldots,p_m;q_1,\ldots,q_m;r_1,\ldots,r_m)}(x_1,\ldots,x_m)$$

$$B_{n_1-k_1,\ldots,n_m-k_m}^{(b';c'_1,\ldots,c'_m;p_1,\ldots,p_m;q_1,\ldots,q_m;r_1,\ldots,r_m)}(x_1,\ldots,x_m).$$

Again making an appeal to generating relation (6.1), we obtain

$$(6.6)\quad B_{n_1,\dots,n_m}^{(b+b';c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=\sum_{k_1=0}^{n_1}\cdots\sum_{k_m=0}^{n_m}\frac{(b')_{k_1+\dots+k_m}\,x_1^{k_1}\dots x_m^{k_m}}{k_1!\dots k_m!}B_{n_1-k_1,\dots,n_m-k_m}^{(b;c_1+r_1k_1,\dots,c_m+r_mk_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m).$$

Further by making an appeal to generating relation (6.1), we establish

$$(6.7)\quad B_{n_1,\dots,n_m}^{(b;c_1+c'_1,\dots,c_m+c'_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=\sum_{k_1=0}^{[n_1/q_1]}\cdots\sum_{k_m=0}^{[n_m/q_m]}\frac{(c_1)_{k_1}}{k_1!}p_1^{k_1}\cdots\frac{(c_m)_{k_m}}{k_m!}p_m^{k_m}\;B_{n_1-q_1k_1,\dots,n_m-q_mk_m}^{(b;c'_1,\dots,c'_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m).$$

7. Differential Recurrence Relations. Differenting (6.1) partially with respect to x_1 and equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$ both the sides, we get

$$(7.1)\quad \frac{\partial}{\partial x_1}B_{n_1,\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=bB_{n_1-1,n_2,\dots,n_m}^{(b+1;c_1+r_1,c_2,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m),$$

which suggests m results in the following unified form:

$$(7.2)\quad \frac{\partial}{\partial x_i}B_{n_1,\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=bB_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_m}^{(b+1;c_1,\dots,c_{i-1},c_i+r_i,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)\;i=1,\dots,m.$$

By an interaction, (7.2) further gives

$$(7.3)\quad \frac{\partial^k}{\partial x_i^k}B_{n_1,\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=(b)_k\,B_{n_1,\dots,n_{i-1},n_i-k,n_{i+1},\dots,n_m}^{(b+k;c_1,\dots,c_{i-1},c_i+kr_i,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m).$$

Now differentiating (6.1) partially with respect to t_1 and equating the coefficients of both sides, we obtain

$$(7.4)\quad c_1p_1q_1B_{n_1-q_1+1,n_2,\dots,n_m}^{(b;c_1+1,c_2,\dots,c_i+kr_i,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$+bx_1B_{n_1,\dots,n_m}^{(b+1;c_1+r,c_2,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$+bx_1p_1(q_1r_1-1)B_{n_1-q_1,n_2,\dots,n_m}^{(b+1;c_1+r,c_2,\dots,c_i+kr_i,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$=B_{n_1+1,n_2,\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m),$$

which suggests m results in the following unified form

$$(7.5)\quad c_i p_i q_i B_{n_1,\dots,n_{i-1},n_i-q_i+1,n_{i+1},\dots,n_m}^{(b;c_1,\dots,c_{i-1},c_i+1,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$+bx_1 B_{n_1,\dots,n_m}^{(b+1;c_1+r,c_2,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$+bx_i p_i(q_i r_i-1) B_{n_1,\dots,n_{i-1},n_i-q_i,n_{i+1},\dots,n_m}^{(b+1;c_1,\dots,c_{i-1},c_i+r_i,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$=B_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m),\quad i=1,\dots,m.$$

Again starting with generating relation (6.1), we derive

$$(7.6)\quad B_{n_1,\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$=B_{n_1,\dots,n_m}^{(b-1;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$-\sum_{i=1}^{m} B_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_m}^{(b-1;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m).$$

That is

$$(7.10)\quad B_{n_1,\dots,n_m}^{(b-1;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$-B_{n_1,\dots,n_m}^{(b;c_1,\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$
$$=\sum_{i=1}^{m} B_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_m}^{(b-1;c_1,\dots,c_{i-1},c_i+r_i,c_{i+1},\dots,c_m;p_1,\dots,p_m;q_1,\dots,q_m;r_1,\dots,r_m)}(x_1,\dots,x_m).$$

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

SUMMABILITY OF FOURIER SERIES BY KARAMATA METHOD

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(Received : September 15,2015)

ABSTRACT

In 1965, Vučković [9] obtained summability of Fourier series by Karamata method K^{λ} and used a condition stronger than continuity which was generalized by Kathal [6] in 1969. In this paper we obtain K^{λ} and K^{λ} $(C,1)$ summability of Fourier series for the different sets of conditions, one of which includes discontinuous functions also.

2010 Mathematics Subject Classification: 42A24, 40G99.
Keywords and phrases: Summability of Fourier series, Summability by product
method, Summability by Karamata method.

1. Introduction. The series $\sum_{n=0}^{\infty} a_n$, with the sequence (s_n) of partial sums, is said to be summable to s by Karamata method $K^{\lambda}, \lambda > 0$, if (Karamata [5])

(1.1) $$s_n^{\lambda} = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m s_m \to s, \text{ as } n \to \infty,$$

where the numbers are defined by

(1.2) $$\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m, \text{ for } \lambda > 0,$$

$n = 0,1,2\ldots; 0 \le m \le n,$ and are absolute values of Stirling numbers of the first kind.

$K^{\lambda}(C,1)$ Summability method is obtained by super-imposing K^{λ}-mean on $(C,1)$ mean of a sequence. Karamata [5] showed that the K^{λ}-

methods are regular for every $\lambda > 0$, therefore $K^{\lambda}(C,1)$ method includes $(C,1)$ method for each $\lambda > 0$.

For real x and s and for any L-integrable function g over $[0,\pi]$, we write

$$(1.3) \quad \varphi(t) = f(x+t) + f(x-t) - 2s,$$

$$(1.4) \quad g_0(t) = g(t),$$

$$(1.5) \quad g_\beta(t) = \frac{1}{t}\int_0^t g_{\beta-1}(u)\,du \ (\beta = 1,2,3),$$

$$(1.6) \quad P(t) = \varphi(t) - \varphi_1(t),$$

$$(1.7) \quad Q(t) = P_1(t) + P_2(t) + P_3(t),$$

$$(1.8) \quad K_n^{\lambda}(t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m \sin(m+1)t, \lambda > 0.$$

$$(1.9) \quad \theta_m(t) = \tan^{-1}\left(\frac{\lambda \sin t}{m + \lambda \cos t}\right), \lambda > 0.$$

Throughout, we use A for a positive constant not necessarily the same at each occurrence.

Let $s_n(f;x)$ denote the partial sum of the first $(n+1)$-terms of the Fourier series of $f \in L_{2\pi}$ at a point x. Then (Titchmarsh [8]; p.403)

$$(1.10) \quad s_n(f;x) - s = \frac{1}{2\pi}\int_0^\pi \frac{\sin(n+1/2)t}{\sin t/2}\varphi(t)\,dt.$$

2. Theorems and comments. In 1957, Agnew [1] proved that the K^{λ}-methods were not Fourier effective for continuous functions. This suggested to consider a stronger condition than continuity to get K^{λ}-summability of Fourier series at a point which was successfully done by Vučković [9] in 1965 and proved the following:

Theorem A. If

$$(2.1) \quad \varphi(t) = o\{1/\log 1/t\}, \text{ as } t \to 0+,$$

then the Fourier series of $f \in L_{2\pi}$ at x is K^{λ}-summable to $s = f(x)$ for every $\lambda > 0$.

Generalizing Theorem A, Kathal [6] gave the following criterion for K^{λ}-summability of the Fourier series:

Theorem B. Let

(2.2) $\int_0^t |\varphi(u)|du = o[t/\log 1/t]$, as $t \to 0+$.

Then the Fourier series of $f \in L_{2\pi}$ at x is K^λ-summable to s for every $\lambda > 0$.

We further mention the following theorem due to Chandra [3] for T-summability of the Fourier series:

Theorem C . Let $C(\varphi)$ denote a non-empty set of conditions involving φ, such that

(2.3) $C(\varphi) \Rightarrow \varphi_1(t) = o(1)$, as $t \to 0+$,

and let T be a regular summability method. If $C(\varphi)$ with (2.3) is a sufficient condition for T-summability of the Fourier series of $f \in L_{2\pi}$ at x to s, then $C(Z)$ along with

(2.4) $\varphi_1(t) = O(1)$, as $t \to 0+$

is also a sufficient condition, where $Z(t) = P(t) + P_1(t)$.

By using Theorem B in Theorem C, we first propose to prove the following theorem which provides larger space of functions $f \in L_{2\pi}$ than the spaces for which Theorems A and B hold:

Theorem 1. Let (2.4) hold and let

(2.5) $\int_0^T |P(u)|du = o[t/\log 1/t]$, as $t \to 0+$.

Then the Fourier series of $f \in L_{2\pi}$ at x is K^λ-summable to s for every $\lambda > 0$.

To observe that the space of functions $f \in L_{2\pi}$ characterised by (2.5) with (2.4) is larger than the spaces characterized by (2.1) or (2.2), we first observe that each of the conditions (2.1) and (2.2) implies (2.5) and (2.4) and to examine that converse is not true, we consider the following example:

Let $x=0$ and $f \in L_{2\pi}$ be even. Then, $\varphi_1(t) = 2f(t)$. Define

(2.6) $f(t) = (\log 2\pi/t)^{-1}$ in $0 < t \le \pi$, and zero for $t = 0$.

Then (2.4) holds and

$$P(t) = \frac{2}{t}\int_0^t (\log 2\pi/u)^{-2} du,$$

which satisfies (2.5). However the function defined in (2.6) neither satisfies (2.1) nor (2.2).

Remark: It is trivial to get Theorems A and B from Theorem 1.

We now propose to replace K^{λ}-summability method by $K^{\lambda}(C,1)$-summability method which is more general than K^{λ}-summability as well as $(C,1)$-summability methods and obtain the following result under weaker conditions than (2.2):

Theorem 2. Let $\varphi(t) \in L(0,\pi)$ be such that

(2.7) $\varphi_2(t) = o(1)$, as $t \to 0+$.

Then the Fourier series of $f \in L_{2\pi}$ at x is $K^{\lambda}(C,1)$ summable to s for every $\lambda > 0$, whenever

(2.8) $\int_0^t |P(u)|\,du = o[t/\log 1/t]$, as $t \to 0+$.

It has been shown in Lemma 5 of this paper that there exists a function $f \in L_{2\pi}$ which does not have limit at the origin but it satisfies the conditions of the above theorem.

3. Lemmas. The following lemmas will be used in the proof of the theorems:

Lemmas 1. (Chandra [4]). Let , $\varphi(t) \in L(0,\pi)$ be such that (2.7) holds. Then

(3.1) $$s_n(f;x) - s = \frac{1}{2}Q(\pi)\cos n\pi + \frac{1}{\pi}\int_0^{\pi} \frac{tQ(t)}{(2\sin t/2)^2}\sin(n+1)t\,dt$$

$$-\frac{(n+1)}{\pi}\int_0^{\pi}\frac{tQ(t)}{2\sin t/2}\cos(n+1/2)t\,dt + o(1), \text{ as } n \to \infty.$$

Lemma 2. Let, for $\lambda > 0$ and $P_m(t) = \left(\lambda^2 + 2\lambda m \cos t + m^2\right)^{1/2}$,

(3.2) $$R(n,t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\prod_{m=0}^{n-1} P_m(t).$$

Then

(3.3) $$K_n^{\lambda}(t) = R(n,t)\sin\left\{t + \sum_{m=1}^{n-1}\theta_m(t)\right\}.$$

Proof. A proof of (3.3) is contained in ([2], Chapter 3; Lemma 1). However, for the convenience of the readers, we are giving its proof.

By using (1.8), we observe that

$$K_n^{\lambda}(t)=\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\operatorname{Im}\left\{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m \exp\big(i(m+1)t\big)\right\},$$

where Im stands for "Imaginary part of" and , by (1.2),

$$\begin{aligned}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m \exp\big(i(m+1)t\big) &= \exp(it)\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\big(\lambda\exp(it)\big)^m \\ &= \exp(it)\prod_{m=0}^{n-1}\big(m+\lambda\exp(it)\big) \\ &= \exp(it)\prod_{m=0}^{n-1}P_m(t)\exp\big(i\theta_m(t)\big) \\ &= \left(\prod_{m=0}^{n-1}P_m(t)\right)\exp\left\{i\left(t+\sum_{m=1}^{n-1}\theta_m(t)\right)\right\}.\end{aligned}$$

Now multiplying the factor $\dfrac{\Gamma(\lambda)}{\Gamma(n+\lambda)}$ on both sides of the above equation and equating the imaginary parts, we get (3.3).

Lemma 3. For every $\lambda = 0$ and $0 < t \le \pi/4$,

(3.4) $$\sin\left\{t+\sum_{m=1}^{n-1}\theta_m(t)\right\}=O(t)\log(n+\lambda).$$

Proof. A proof of (3.4) is contained in ([7], Lemma 5.6). However for the ready reference, we give its proof.

We observe that

(3.5) $$\left|\sin\left\{t+\sum_{m=1}^{n-1}\theta_m(t)\right\}-\sin l(n)t\right| \le \left|t+\sum_{m=1}^{n-1}\theta_m(t)-l(n)t\right|,$$

where

(3.6) $$l(n)=1+\lambda\sum_{m=1}^{n-1}\frac{1}{m+\lambda}=O(1)\log(n+\lambda).$$

We further observe that

(3.7) $$0<\frac{\lambda\sin t}{m+\lambda\cos t}<1, \text{ for } 0<t\le\pi/4 \text{ and } m\ge 1.$$

Therefore for $0<t\le\pi/4$, we get

$$(3.8)\ \theta_m(t)=\left[\tan^{-1}\left(\frac{\lambda\sin t}{m+\lambda\cos t}\right)-\frac{\lambda\sin t}{m+\lambda\cos t}\right]+\left[\frac{\lambda\sin t}{m+\lambda\cos t}-\frac{\lambda t}{m+\lambda\cos t}\right]$$

$$+\left[\frac{\lambda t}{m+\lambda\cos t}-\frac{\lambda t}{m+\lambda}\right]+\frac{\lambda t}{m+\lambda}$$

$$=O\left\{\left(\frac{\lambda\sin t}{m+\lambda\cos t}\right)^3\right\}+O\left\{\frac{t^3}{m+\lambda\cos t}\right\}+O\left\{\frac{t^3}{(m+\lambda)(m+\lambda\cos t)}\right\}+\frac{\lambda t}{m+\lambda}$$

$$=\frac{\lambda t}{m+\lambda}+O\left\{\frac{t^3}{m}\right\},1\le m\le n-1.$$

Using (3.8), we get

$$(3.9)\ t+\sum_{m=1}^{n-1}\theta_m(t)=t\left\{1+\lambda\sum_{m=1}^{n-1}\frac{1}{m+\lambda}\right\}+O(t^3)\sum_{m=1}^{n-1}\frac{1}{m}$$

$$=tl(n)+O(t^3)\log n.$$

Using (3.9) in (3.5), we get (3.4).

Lemma 4. For $0<t<\pi/2$,

$$(3.10)\ K_n^{\lambda}(t)=O\left[\exp\left\{-(\lambda/3)t^2\log n\right\}\right].$$

Proof. We observe that

$$\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\lambda^m\sin(m+1)t=\operatorname{Im}\left\{\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\lambda^m\exp\{i(m+1)t\}\right\}$$

$$=\operatorname{Im}\left\{\exp(it)\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(\lambda\exp(it))^m\right\}$$

$$=\operatorname{Im}\left\{\exp(it)\frac{\Gamma(n+\lambda\exp(it))}{\Gamma(\lambda\exp(it))}\right\},$$

by (1.1). Hence for some positive constant A, we get

$$(3.11)\ \left|K_n^{\lambda}(t)\right|\le A\frac{\Gamma(n+\lambda\cos t)}{\Gamma(n+\lambda)}$$

and for $\lambda>0$

$$(3.12)\ \frac{\Gamma(n+\lambda\cos t)}{\Gamma(n+\lambda)}=\frac{\Gamma(\lambda\cos t)}{\Gamma(\lambda)}\frac{\binom{n+\lambda\cos t-1}{n}}{\binom{n+\lambda-1}{n}}$$

$$\sim n^{-\lambda(1-\cos t)} = \exp\{-\lambda(1-\cos t)\log n\}.$$

Also for $0 < t < \pi/2$,

(3.13) $1-\cos t > t^2/3$.

Using (3.12) and (3.13) in (3.11), we get (3.10).

Lemma 5. Let (2.2) holds. Then

(3.14) $\varphi_1(t) \in L(0,\pi)$

(3.15) $\varphi_2(t) = o(1)$, as $t \to 0+$

(3.16) $\int_0^t |P_1(u)|du = o\{t/\log 1/t\}$, as $t \to 0+$.

However the converse is not true in general.

Proof. We first observe that (2.2) implies that (3.14) and (3.15) hold. Also (2.2) implies that

$$\int_0^t |\varphi_1(u)|du = o\{t/\log 1/t\}, \text{ as } t \to 0+.$$

which together with (2.2) implies that

$$\int_0^t |P(u)|du = o\{t/\log 1/t\}, \text{ as } t \to 0+,$$

which further implies (3.16).

To prove that the converse is not true in general, we consider an even function $f \in L_{2\pi}$ and the point $x=0$. Then for $s=0$, $\varphi_1(t) = 2f(t)$. We now define

(3.17) $f(t) = 2t\sin 1/t - \cos 1/t$ in $(0,\pi)$.

Then

$$\varphi_1(t) = \frac{2}{t}\int_0^t f(u)\,du = \frac{2}{t}\int_0^t \frac{d}{du}\left(u^2 \sin 1/u\right)du = 2t\sin 1/t$$

and hence all the conditions: (3.14), (3.15) and (3.16) are satisfied by the function f defined in (3.17). However

$$\begin{aligned}\int_0^t |\varphi(u)|du &= 2\int_0^t |2u\sin 1/u - \cos 1/u|du \\ &\geq 2\int_0^t |\cos 1/u|du - 4\int_0^t u|\sin 1/u|du \\ &\geq 2\int_0^t |\cos 1/u|du - 2t^2 \\ &\geq \int_0^t 2\cos 1/u\, du - 2t^2\end{aligned}$$

$$\geq \int_0^t (1+\cos 2/u)\,du - 2t^2$$

$$= t + \int_0^t \cos 2/u\; du - 2t^2.$$

Hence

$$\int_0^t |\varphi(u)|\,du \neq o\{t/\log 1/t\}, \text{ as } t \to 0+.$$

Observe that the function defined in (3.17) is not continuous at t=0. This completes the proof of the lemma.

4. Proof of the Theorems

4.1. Proof of Theorem 1. In Theorem C, we take T to be K^{λ} summability method, which is regular and suppose $C(\varphi)$ is a set containing one condition (2.2), which involves φ. Then (2.3) and (2.4) hold and by using Theorem B in Theorem C, we obtain that

(4.1) $\int_0^t |Z(u)|\,du = o\{t/\log 1/t\}$, as $t \to 0+$.

Is a sufficient condition for K^{λ}-summability of Fourier series of $f \in L_{2\pi}$ at a point x. Now, we use the inverse formula :

(4.2) $P(u) = Z(u) - u^{-2}\int_0^u yZ(y)\,dy,$

which may be easily verified by substituting the definition of Z and simplifying the expression on the right.

Integrating and changing the order of integration in the second integral on the right of (4.2), we get

(4.3) $\int_0^t |P(u)|\,du < 2\int_0^t |Z(u)|\,du = 0\{t/\log 1/t\}$, as $t \to 0+$,

by (4.1). This proves that (2.5) along with (2.4) is a sufficient condition.

This completes the proof of Theorem 1.

4.2. Proof of Theorem 2. We observe that (2.8) implies

(4.4) $\int_0^t |Q(u)|\,du = o\{t/\log 1/t\}$, as $t \to 0+$,

therefore we use (4.4) in place of (2.8) for the proof of the theorem.

Let $T_n(f;x)$ and $t_n(f;x)$ denote respectively $K^{\lambda}(C,1)$ and $(C,1)$ means of $s_n(f;x)$. Then , by (1.1),

$$T_n(f;x) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m t_m(f;x)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m \{t_m(f;x) - s\} + s$$

and hence

$$(4.5)\ T_n(f;x) - s = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m \{t_m(f;x) - s\},$$

where, by Lemma 1,

$$(4.6)\quad t_m(f;x) - s = \frac{1}{m+1} \sum_{v=0}^{m} (s_v(f;x) - s)$$

$$= \frac{1}{2} Q(\pi) \frac{1}{m+1} \sum_{v=0}^{m} \cos v\pi + \frac{1}{\pi} \frac{1}{m+1} \int_0^{\pi} \frac{tQ(t)}{(2\sin t/2)^2} \left(\sum_{v=0}^{m} \sin(v+1)t \right) dt$$

$$- \frac{1}{\pi} \int_0^{\pi} \frac{tQ(t)}{2\sin t/2} \left(\sum_{v=0}^{m} \cos(v+1/2)t \right) dt + o(1)$$

$$= I_m^{(1)} + I_m^{(2)} + I_m^{(3)} + o(1),\ \text{say, as}\ m \to \infty.$$

It is obvious that

$$(4.7)\ I_m^{(1)} = o(1),\ \text{as}\ m \to \infty.$$

Now to prove that $I_m^{(2)} = o(1)$, as $m \to \infty$, we write

$$(4.8)\ \pi I_m^{(2)} = \frac{1}{m+1} \left(\int_0^{\frac{1}{m+1}} + \int_{\frac{1}{m+1}}^{\pi} \right) \left(\frac{tQ(t)}{(2\sin t/2)} \left(\sum_{v=0}^{m} \sin(v+1)t \right) dt \right) = J_m^{(1)} + J_m^{(2)},$$

say.

By using $|\sin(v+1)t| \le (v+1)t$ and (4.4), we can easily get

$$(4.9)\ J_m^{(1)} = o(1),\ \text{as}\ m \to \infty,$$

and by using

$\sum_{v=0}^{m} \sin(v+1)t = O(t^{-1})$, uniformly in $0 < t < \pi$ in $J_m^{(2)}$,

integrating by parts and appealing (4.4), we get

$$(4.10)\ J_m^{(2)} = O\left(\frac{1}{m+1} \right) \int_{1/(m+1)}^{\pi} |Q(t)| t^{-2} dt$$

$$= O\left\{\frac{1}{\log(m+1)}\right\} + O\left\{\frac{1}{m+1}\right\}\int_{1/(m+1)}^{\pi} t^{-2}\left(\log\frac{1}{t}\right)^{-1} dt$$

$$= O\left\{\frac{1}{\log(m+1)}\right\} = o(1), \text{ as } m \to \infty.$$

Thus by using (4.9) and (4.10) in (4.8), one gets

(4.11) $I_m^{(2)} = o(1)$, as $m \to \infty$.

and collecting (4.6), (4.7) and (4.11), we get

(4.12) $t_m(f;x) - s = I_m^{(3)} + o(1)$, as $m \to \infty$,

where , for $0 < \delta < \pi/4$,

$$I_m^{(3)} = -\frac{1}{\pi}\int_0^{\pi}\frac{tQ(t)}{(2\sin t/2)^2}\sin(m+1)t\, dt$$

$$= -\frac{1}{\pi}\int_0^{\pi}\frac{tQ(t)}{(2\sin t/2)^2}\sin(m+1)t\, dt + o(1),$$

by using

$$\sum_{v=0}^{m}\cos(v+1/2)t = \frac{\sin(m+1)t}{2\sin t/2}$$

and Riemann-Lebesgue theorem. Further, we observe that

$$\left\{\frac{t}{2\sin t/2} - 1\right\}\frac{Q(t)}{2\sin t/2} \in L(0,\delta) .$$

Therefore by using once again Riemann-Lebesgue theorem, we get

(4.13) $I_m^3 = -\dfrac{1}{\pi}\displaystyle\int_0^{\delta}\frac{Q(t)}{2\sin t/2}\sin(m+1)t\, dt + o(1)$, as $m \to \infty$.

Now, combining (4.12) and (4.13), we get

(4.14) $t_m(f;x) - s = -\dfrac{1}{\pi}\displaystyle\int_0^{\delta}\frac{Q(t)}{2\sin 1/2}\sin(m+1)t\, dt + o(1)$, as $m \to \infty$.

By using the regularity of K^{λ}-mean and (4.14) in (4.5), we get

$$T_m(f;x) - s = -\frac{1}{\pi}\int_0^{\delta}\frac{Q(t)}{2\sin t/2}K_n^{\lambda}(t)\,dt + o(1) = R_n + o(1), \text{ say.}$$

Thus for the proof of the theorem, it is sufficient to prove that

(4.15) $R_n = o(1)$, as $m \to \infty$.

For, $a_n = \dfrac{1}{\log(n+\lambda)}$ and $b_n = \alpha_n^{\alpha}$, where $\lambda > 0$ and $0 < \alpha < 1/2$, we write

(4.16) $$R_n = -\frac{1}{\pi}\left(\int_0^{a_n} + \int_{a_n}^{b_n} + \int_{b_n}^{\delta}\right)\left(\frac{Q(t)}{2\sin t/2}K_n^{\lambda}(t)\,dt\right) = R_{n,1} + R_{n,2} + R_{n,3}, \text{ say.}$$

Now, from Lemma 2, we have

(4.17) $$K_n^{\lambda}(t) = R(n,t)\sin\left\{t + \sum_{m=1}^{n-1}\theta_m(t)\right\},$$

where

(4.18) $$R(n,t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\prod_{m=0}^{n-1}P_m(t) \le 1,$$

since

(4.19) $$P_m(t) = \left(\lambda^2 + 2\lambda m\cos t + m^2\right)^{1/2} \le P_m(0) \le (m+\lambda).$$

Using (4.18) and Lemma 3 in (4.17), we get

(4.20) $$K_n^{\lambda}(t) = O(t)\log(n+\lambda).$$

By using (4.20), we get

(4.21) $$R_{n,1} = O(1)\log(n+\lambda)\int_0^{a_n}|Q(t)|\,dt = o(1), \text{ by (4.4), as } n\to\infty.$$

Also by (4.18), we observe that

(4.22) $$\left|K_n^{\lambda}(t)\right| \le 1,$$

therefore by using (4.22) , we get

(4.23) $$R_{n,2} = O(1)\int_{a_n}^{b_n}t^{-1}|Q(t)|\,dt$$

$$= o(1) + o(1)\int_{a_n}^{b_n}t^{-1}(\log 1/t)^{-1}\,dt = o(1), \text{ by (4.4), as } n\to\infty.$$

Finally by Lemma 4, we get

(4.24) $$R_{n,3} = O(1)\int_{b_n}^{\delta}t^{-1}|Q(t)|\exp\left\{-\frac{\lambda}{3}t^2\log n\right\}dt$$

$$= O(1)\exp\left\{-\frac{\lambda}{3}b_n^2\log n\right\}\int_{b_n}^{\delta}t^{-1}|Q(t)|\,dt$$

and integrating by parts and using (4.4), we get

(4.25) $$\int_{b_n}^{\delta}t^{-1}|Q(t)|\,dt = O(1) + O(1)\int_{b_n}^{\delta}\frac{1}{t\log 1/t}\,dt = O\{\log\log\log n\}.$$

Using (4.25) in (4.24), we get

(4.26) $R_{n,3} = o(1)$, as $n \to \infty$.

Using (4.21), (4.23) and (4.26) in (4.16), we get (4.15), i.e.,

$R_n = o(1)$, as $n \to \infty$.

This completes the proof of Theorem 2.

We now state the following corollary of Theorem 2:

Corollary. Let, for $0 < \delta \leq \pi, \varphi_1(t) \in L(0,\delta)$ be such that

(4.34) $\int_0^t |\varphi_1(u)| du = o\{t/\log 1/t\}$, as $t \to 0+$.

Then the Fourier series of $f \in L_{2\pi}$ at x is $K^\lambda(C,1)$ summable to s for every $\lambda > 0$.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

ON TOPOLOGICAL NEAR ALGEBRAS

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(Received :July 24, 2015; *Revised : October* 24, 2015)

ABSTRACT

In this paper we address the problems of completion of a normed near algebra, normability of a topological near algebra and representation of a normed near algebra. While the results about completion and normability proceed along predictable lines, the representation theorem makes use of ideas not familiar to many and we feel that it deserves the notice of the mathematical community.

2010 Mathematics Subject Classification : 16Y30.

Keywords and Phrases: Near Algebras, Semilinear Transformation, Topological Near Algebra.

1. Preliminaries and Introduction. *G.* Pilz says in the preface to his book on near rings [3] that one might view ring theory as the "linear theory of group mappings", while near rings provide the "nonlinear theory". 'Surprisingly', a lot of "linear results" can be transferred to the general case after suitable changes. We stress the word 'Surprisingly' in the quote. The transition from linear to the nonlinear case is not an obvious and smooth process but involves a huge leap in imagination and intuition. Stanislaw Ulam is reported to have quipped that "Studying nonlinear dynamics is like studying nonelephant Biology" to drive home the point that nonlinearity is pervasive and linearity is mainly a mathematical tool to

simplify matters to an easy level of mathematical manoeuvrability. The hopes of handling nonlinear problems directly without approximating them with linear problems were kindled by the iterative techniques of analysis and the algebraic techniques of near rings, near algebras and so on. The study of topological near algebras [4], topological near rings etc. is motivated by the desire to address nonlinear questions by combining the algebraic and analytical structures, particularly suited for the study of nonlinear problems.

Let us recall that a metric space is complete if and only if every Cauchy sequence is convergent. By a completion of a metric space (X,d) we mean a metric space $\left(\tilde{X},\tilde{d}\right)$ which satisfies:

1. $\left(\tilde{X},\tilde{d}\right)$ is complete,
2. There is an isometry $f:(X,d)\rightarrow\left(\tilde{X},\tilde{d}\right)$ such that $f(X)$ is dense in $\tilde{X}$.

The completion of a complete metric space is itself complete and any two completions of a metric space are isometrically isomorphic. We also know that the completion of a normed linear space is not only a complete metric space but also a complete normed linear space.

2. Completion of a Normed Near Algebra. Let us recall the following definitions:

Definition 2.1. [6] A (right) near algebra V over a field $\mathbb{R}$ is a linear space over $\mathbb{R}$ on which multiplication '*' is defined such that

1. V forms a semi group under multiplication,
2. multiplication is right distributive w.r.t addition: $(x+y)*z=x*z+y*z$ and
3. $\alpha(x*y)=(\alpha x)*y$ for all $x,y\in V$ and $\alpha\in R$.

Example 2.2. Let V be a linear space over the field R and $\varphi:V\rightarrow\mathbb{R}$ be a mapping such that $\phi(\phi(u)v)=\phi(u)\phi(v)$ for all $u,v\in V$. Now define '*' 0 on V as $a*b=\phi(b)a$ for all $a,b\in V$. Then $(V,+,\cdot,*)$ is a right near algebra.

Definition 2.3. Let X be a linear space. A mapping $R:X\rightarrow End(X)$ is said to be a semilinear transformation if for every v,w in X, $R(R(w)v)=R(w)\circ R(v)$. From now we write R_w for $R(w)$.

Theorem 2.4 [4]. Suppose X is a linear space and R is semilinear transformation on X. Define $x*y=R_y(x)$ for all x, $y\in X$. Then $(X,+,\cdot,*)$ is a near algebra. Conversely if $(X,+,\cdot,*)$ is a near algebra, define for $x\in X$, $R_x:X\rightarrow X$ by $R_x(y)=y*x$. Then R is a semilinear transformation on X. Also $(X,+,\cdot,*)$ is an algebra if and only if R is linear.

Definition 2.5. A normed near algebra is a near algebra $(X,+,\cdot,*)$ together with a norm w.r.t which X is a normed linear space such that $\|x*y\| \le \|x\|\|y\|$ holds for $x,y \in X$.

In addition if the norm satisfies $\|x*y - x*z\| \le \|x\|\|y-z\|$ for all $x,y,z \in X$, then we say X is modular.

Example 2.6. Let X be a normed linear space and $\phi : X \to \mathbb{R}$ be a mapping such that $\phi(\phi(u)v) = \phi(u)\phi(v)$ for all $u,v \in X$ and $|\phi(v)| \le \|v\|$ for all $v \in X$. Then by example 2.2 $(X,+,\cdot,*)$ is a near algebra. Now $\|x*y\| = \|\phi(y)x\| = |\phi(y)|\|x\| \le \|y\|\|x\| = \|x\|\|y\|$. So $(X,+,\cdot,*)$ is a normed near algebra.

Lemma 2.7. Let $(X,+,\cdot,*)$ be a normed near algebra $\{x_n\},\{x'_n\},\{y_n\},\{y'_n\}$ be bounded sequences in X such that $\lim(x_n - x_{n'}) = 0$ and $\lim(y_n - y'_n) = 0$. Then $\lim[(x_n * y_n) - (x'_n * y'_n)] = 0$ provided $\|x*y - x*z\| \le \|x\|\|y-z\|$ for all $x,y,z \in X$.

Proof. $\|x_n * y_n - x'_n * y_n\| = \|x_n * y_n - x'_n * y_n + x' * y_n - x'_n * y'_n\|$

$$\le \|x_n * y_n - x'_n * y_n\| + \|x'_n * y_n - x'_n * y'_n\| \le \|x_n - x'_n\|\|y_n\| + \|x'_n\|\|y_n - y'_n\|$$

Since $\{x'_n\}$ and $\{y_n\}$ are bounded and $\lim\|x_n - x'_n\| = \lim\|y_n - y'_n\| = 0$ it follows that $\lim[(x_n * y_n) - (x'_n * y'_n)] = 0$.

Theorem 2.8. The completion of a modular normed near algebra $(X,+,\cdot,*)$ is a complete modular normed near algebra. Further if the multiplication $*$ in X is induced by R as in Theorem 2.4 then the multiplication in $\tilde{X}$ is induced by $\tilde{R} : \tilde{X} \to \mathbb{R}$ defined by $\tilde{R}_{\tilde{y}}(\tilde{x}) = \lim R_{y_n}(x_n)$ for each $\tilde{y} \in \tilde{X}$ with $\lim y_n = \tilde{y}$, then for $\tilde{y} \in \tilde{X}$ and $\tilde{y} = \lim y_n$, $\tilde{R}_{\tilde{y}} = \lim_{n\to\infty} R_{y_n}$.

Proof. We know that the Banach space $\tilde{X}$ denote the completion of the normed linear space X [4]. Define $*$ on $\tilde{X}$ by $\tilde{x} * \tilde{y} = \lim(x_n * y_n)$ where $x_n, y_n \in X$ for all n and $\lim x_n = \tilde{x}, \lim y_n = \tilde{y}$.

This definition is meaningful by Lemma 2.7.

To show that $*$ is continuous.

Suppose $x_n \to \tilde{x}$ and $y_n \to \tilde{y}$. Then

$$
\begin{aligned}
0 &\le \|x_n * y_n - \tilde{x} * \tilde{y}\| \\
&= \|x_n * y_n - x_n * \tilde{y} + x_n * \tilde{y} - \tilde{x} * y\| \le \|x_n * y_n - x_n * \tilde{y}\| + \|x_n * \tilde{y} - \tilde{x} * \tilde{y}\| \\
&\le \|x_n\|\|y_n - \tilde{y}\| + \|x_n - \tilde{y}\|\|\tilde{y}\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore '$*$' is continuous.

Let $\tilde{x},\tilde{y},\tilde{z} \in \tilde{X}$. Then $\tilde{x}*(\tilde{y}*\tilde{z}) = \tilde{x}*\lim(y_n * z_n) = \lim x_n * \lim(y_n * z_n) =$

$\lim(x_n *(y_n * z_n)) = \lim((x_n * y_n) * z_n) = \lim(x_n * y_n) * \lim z_n = (\tilde{x}*\tilde{y})*\tilde{z}.$

Also

$(\tilde{x}+\tilde{y})*\tilde{z} = \lim(x_n + y_n)*z_n = \lim[(x_n * z_n)+(y_n * z_n)] = \lim(x_n * z_n) + \lim(y_n * z_n) =$

$(\tilde{x}*\tilde{z})+(\tilde{y}*\tilde{z})$. Therefore $\tilde{X}$ is a normed near algebra which is complete.

Further if $\tilde{x},\tilde{y},\tilde{z} \in \tilde{X}, \tilde{x} = \lim x_n, \tilde{y} = \lim y_n$ and $\tilde{z} = \lim z_n$ where $x_n, y_n, z_n \in X$ for all n then for every n, $\|x_n * y_n - x_n * z_n\| \le \|x_n\|\|y_n - z_n\|$ for all n.

By taking limit as $n \to \infty$, we get

$\|\tilde{x}*\tilde{y} - \tilde{x}*\tilde{z}\| \le \|\tilde{x}\|\|\tilde{y}-\tilde{z}\|$.

Hence $\tilde{X}$ is a complete modular normed near algebra.

Suppose $\tilde{x}\tilde{y} \in \tilde{X}$ and $\{x_n\},\{y_n\}$ be sequences in X such that $\tilde{x} = \lim x_n$, $\tilde{y} = \lim y_n$. Define $\tilde{R}_{\tilde{y}} = \lim R_{y_n} x_n$.

This definition is of course meaningful.

Further we can verify that

(1) $\tilde{R}_{\tilde{x}}(\alpha\tilde{z}) = \alpha\tilde{R}_{\tilde{x}}(\tilde{z})$,

(2) $\tilde{R}_{\tilde{R}_{\tilde{x}(\tilde{y})}}(\tilde{z}) = \tilde{R}_{\tilde{x}}(\tilde{R}_{\tilde{y}}(\tilde{z}))$ for all $\tilde{x},\tilde{y},\tilde{z} \in \tilde{X}$ and $\alpha \in$ R.

Remark 2.9. Let X be the normed linear space of all Lipschitz functions f on $\mathbb{R}$ with $f(0) = 0$. Then X is a normed near algebra with respect to the multiplication defined as the composition of maps and

$$\|f\|_L = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} / x,y \in X, x \ne y\right\}.$$

Then X is complete. However the condition $\|fog - foh\|_L \le \|f\|_L \|g-h\|_L$ does not necessarily hold good as is evident from the following:

Let f,g and h from $\mathbb{R}$ into $\mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x+1 & if\,|x|>1 \\ 0 & if\,-1 \le x \le 1 \end{cases},$$

$$g(x) = x \text{ for all } x \in \mathbb{R},$$

$$h(x) = 2x \text{ for all } x \in \mathbb{R}.$$

$$\text{Now } [fog - foh](x) = f(x) - f(2x) = \begin{cases} -x & \text{if } |x| > 1 \\ -2x - 1 & \text{if } -1 \le x < -1/2 \\ 0 & \text{if } -1/2 \le x \le 1/2 \\ 1 - 2x & \text{if } 1/2 < x \le 1. \end{cases}$$

It is evident that $fog - foh$ is differentiable at all but finitely many (exactly five) points and hence can be realized almost everywhere as the Riemann(=Lebesgue) integral of its derivative extended to the whole of $\mathbb{R}$ by defining arbitrarily at the exceptional five points. Hence $fog - foh$ is absolutely continuous.

$$\text{Futrher } (fog - foh)'(x) = \begin{cases} -1 & \text{on } (-\infty, -1) \cup (1, \infty) \\ -2 & \text{on } (-1, -1/2) \cup (1/2, 1) \\ 0 & \text{on } (-1/2, 1/2) \end{cases}$$

so that $\left|(fog - foh)'(x)\right| \le 2$. Hence [5] $fog - foh \in X$ and $\|fog - foh\|_L = 2$.

$$\text{Also } f'(x) = \theta \begin{cases} 1 & \text{on } (-\infty, -1) \cup (1, \infty) \\ 1 & \text{on } (-1, 1). \end{cases}$$

So $\|f\|_L = 1$.

Also $(g - h)(x) = -x$ for all x and $(g - h)'(x) = -1$ for all $x \in \mathbb{R}$ and so $\|g - h\|_L = 1$.

So $2 = \|fog - foh\|_L \le 1.1 = \|f\|_L \|g - h\|_L$.

Theorem 2.10. Let X be the normed linear space of all Lipschitz functions f on $\mathbb{R}$ with $f(0) = 0$. Then L_X is a complete normed near algebra.

Proof. From [1] L_X is a normed near algebra.

Let $\{f_n\}$ be a Cauchy sequence. Then for every $\varepsilon > 0$ there exists N such that $\|f_n - f_m\|_L < \varepsilon$ for $n > m \ge N$.

For $x \neq y$, $\dfrac{\left|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))\right|}{|x - y|} < \varepsilon$ for $n > m \ge N$.

In particular for $x \neq 0$,

$$\frac{|f_n(x) - f_m(x)|}{|x|} < \varepsilon \text{ for } n > m \ge N. \text{ [Since } f_n(0) = f_m(0) = 0\text{]}.$$

Hence $\{f_n(x)\}$ is a Cauchy sequence for every x.

Let $f(x) = \lim f_n(x)$ for $x \neq 0$ and $f(0)=0$.

We show that $f \in L_X$ and $\lim f_n = f$ in L_X.

Taking $\varepsilon > 0$ there exists N such that

$$\frac{\left|\left(f_n(x)-f_n(y)\right)-\left(f_N(x)-f_N(y)\right)\right|}{|x-y|}<1 \text{ for } n \geq N$$

$$\Rightarrow \frac{\left|f_n(x)-f_n(y)\right|}{|x-y|}<1+\frac{\left|f_N(x)-f_N(y)\right|}{|x-y|} \quad \text{for } n \geq N \text{ and } x \neq y.$$

Since $f_N \in L_X$, there exists $k>0$ such that

$$\frac{\left|f_N(x)-f_N(y)\right|}{|x-y|} \leq k \text{ for } n \geq N \text{ and } x \neq y.$$

Letting $n \to \infty$, we get

$$\frac{|f(x)-f(y)|}{|x-y|} \leq 1+k \ (\text{for } x \neq y) \Rightarrow |f(x)-f(y)| \leq (1+k)|x-y|.$$

So $f \in L_X$.

To show that $f = \lim f_n$ in L_X.

Let $\varepsilon > 0$. Then there exists N_ε such that for $n>m>N_\varepsilon$,

$$\frac{\left|\left(f_n(x)-f_n(y)\right)-\left(f_m(x)-f_m(y)\right)\right|}{|x-y|}<\varepsilon \quad \text{whenever } n>m \geq N_\varepsilon \text{ and } x \neq y.$$

Fix m. Letting $n \to \infty$, we get

$$\frac{\left|\left(f(x)-f(y)\right)-\left(f_m(x)-f_m(y)\right)\right|}{|x-y|} \leq \varepsilon \text{ for } m \geq N_\varepsilon$$

Hence $\lim f_m = f$ in L_X.

This shows that L_X is complete.

3. Normability. In this section we extend an important theorem in topological vector spaces [2]. Topological near algebras, normability and boundedness of topological near algebras are introduced and studied. Using them we derive a sufficient condition for a topological near algebra to be normable.

Definition 3.1 A topological near algebra is a system $(X,+,\cdot,*,\tau)$ satisfying:

(1) $(X,+,\cdot,*)$ is a near algebra,

(2) (X,τ) is a topological space,

(3) the functions

- $(x, y) \to x + y$ of $X \times X$ into X,
- $(a,x) \to ax$ of $\mathbb{R} \times X$ into X and
- $(x, y) \to x*y$ of $X \times X$ into X are continuous and
- if $x \neq 0$, there is an open set U such that $0 \in U$ and $x \notin U$.

Remark 3.2. Let $(X,+,\cdot,*)$ be a modular normed near algebra and τ be the topology induced by the norm. Then $(X,+,\cdot,*,\tau)$ is a topological near algebra.

Definition 3.3. A topological near algebra is said to be *normable* if its topology is generated by a norm.

Definition 3.4. Let X be a topological near algebra. A set $A\subseteq X$ is said to be *bounded* if for every neighborhood U of 0 there is a scalar a such that $A\subseteq aU$.

Remark 3.5. In a normed near algebra X, a set $A\subseteq X$ is bounded in the sense of the above definition if and only if it is bounded in the usual sense.

Theorem 3.6. A topological near algebra X is normable if and only if there is a nonempty set $A\subseteq X$ which is open, bounded, convex, and such that $A*A\subseteq A$.

Proof. Our proof goes on the lines of Theorem[2]. Let X be normable and let $\|\cdot\|$ be the norm in X which yields the given topology.

Clearly $U=\{x\in X/\ \|x\|<1\}$ is nonempty and open.

Let G be a neighborhood of $0\exists a>0\ni U\subseteq aG$ so that U is bounded.

We can easily show that U is convex.

Now if $x\in U$ then $x*x\in U$ since now $\|x*x\|\le\|x\|\|x\|<1$. That $U*U\subset U$ follws from inequality $\|x*y\|\le\|x\|\|y\|$. So $U*U\subseteq U$.

Conversely suppose that X contains a nonempty, open, bounded, convex set H and $H*H\subseteq H$.

If $x\in H, U=H-x=\{y/y=z-x, x\in H\}$ is nonempty and open.

If W is any neighborhood of 0 such that $W-W\subseteq G$,

since H is bounded there exists a real number a such that $H\subseteq aW$ so that $H-x\subseteq H-H\subseteq aW-aW\subseteq a(W-W)\subseteq aG.$

Hence U is bounded.

Convexity of U follows from that of H.

Thus $V=U\cap(-U)$ is non empty, open, bounded, convex and symmetric.

As in [4] we can show that for $x\neq 0$,

the set $E_x=\{a/a\ge 0, x\notin aV\}$ is bounded above.

(2) Define $\|x\|=\sup E_x$

If $x\neq 0$, then there exists a neighborhood G of 0 such that $x\notin G$.

Since V is bounded there exists $a>0$ such that $V\subseteq aG$ so that $x\notin\frac{1}{a}V$

$\Rightarrow\|x\|\ge\frac{1}{a}>0$. Hence $\|x\|=0\Rightarrow x=0.$

Since $\|x\|=\|-x\|$ to prove that $b\|x\|=\|bx\|$ it is enough to prove that

(3) if $b>0$, $x\neq 0$ and $x\in aV$ then $bx\in baV$.

$\Rightarrow \|bx\|\leq ba \Rightarrow \frac{\|bx\|}{b}\leq a$ for all $a \ni x\in aV \Rightarrow \|bx\|\leq b\|x\| \Rightarrow \|bx\|=b\|x\|$.

(4) The proof for $\|x+y\|\leq\|x\|+\|y\|$ is routine.

(5) To show that $\|x*y\|\leq\|x\|\|y\|$.

If $\alpha>1$, then $\|x\|\sqrt{\alpha}>\|x\|. \Rightarrow x\in\|x\|\sqrt{\alpha}V$ Similarly $y\in\|y\|\sqrt{\alpha}V$. Thus

$x*y\in\|x\|\|y\|\alpha V*V\subseteq\|x\|\|y\|\alpha V\Rightarrow x*y\leq\|x\|\|y\|\alpha$

This is true for all $\alpha>1$.

Hence $\|x*y\|\leq\|x\|\|y\|$.

We show that the topology defined by this norm agrees with the given topology.

If G is any neighborhood of 0, there is $a>0$ such that $aV\subseteq G$.(since V is bounded) If $\|x\|<a$, then $x\in aV\subseteq G$. Therefore $\{x/\|x\|<a\}\subseteq G$.

Suppose U is any nonempty open set and $x\in U$.

Then $U-x$ is a neighborhood of 0.

Then there is $r>0$ such that $\{y/\|y\|<r\}\subseteq U-x$.

Therefore U is open in the norm topology.

Thus every open set in X is open in the norm topology.

Conversely we show that every open sphere with center 0 is an open set.

Let $S_r(0)$ be an open sphere, $x\in S_r(0)$ and $\|x\|<t<r$. Then $x\in tU$

If $x\in S_r(0)$, then $\|x\|<r$. Choose t such that $\|x\|<t<r$. Since $\|x\|<t$, $x\in tU$.

If $y\in tU$, then $\|y\|<t$ implies $\|y\|<r$ so that $y\in S_r(0)$. Hence $tU\subset S_r(0)$.

Since tU is open, $S_r(0)$ is open.

If $S_r(x)$ is any open sphere, then $S_r(x)=S_r(0)+x$ and so $S_r(x)$ is open.

Hence every open set in the norm topology is open in X.

4. Embedding. In this section we study the embedding of a normed near algebra in a normed near algebra with better properties and show that any normed near algebra X can be embedded in L_{L_X} where L_X has the meaning given in [1].

Definition 4.1. We say that a near algebra $(X,+,\cdot,*)$ can be embedded in a near algebra $(Y,+,\cdot,\diamond)$ if there exists a near algebra monomorphism from X into Y.

i.e., a mapping $\phi: X\rightarrow Y$ such that

1. ϕ is one-one,

2. $\phi(x+y)=\phi(x)+\phi(y)$
3. $\phi(\alpha x)=\alpha\phi(x)$
4. $\phi(x*y)=\phi(x)\diamond\phi(y)$ for all $x,y\in X,\alpha\in\mathbb{R}$.

If X and Y are normed near algebras, we say that X can be embedded in Y if there exists a near algebra monomorphism $\phi : X\rightarrow Y$ such that $\|\phi(x)\|=\|x\|$ for $x\in X$.

Theorem 4.2. Every normed near algebra X can be embedded in the normed near algebra of zero preserving Lipschitz functions on the normed linear space L_X.

Proof. Let X be a normed near algebra and L_X be the normed near algebra of zero preserving Lipschitz functions on X equipped with the norm defined by

$$\|f\|_L=\sup\left\{\frac{|f(x)-f(y)|}{|x-y|}/\,x,y\in X,x\neq y\right\}.$$

For $x\in X$, define $\hat{x}$ on X by $\hat{x}(y)=y*x$. Clearly $\hat{x}(0)=0$.

Now $\|\hat{x}(y)-\hat{x}(z)\|=\|y*x-z*x\|=\|(y-z)x\|\leq\|y-z\|\|x\|$.

$\hat{x}\in L_X$ and $\|\hat{x}\|\leq\|x\|$.

Further $\|\hat{x}(x)\|=\|x*x\|\leq\|x\|^2\Rightarrow$ if $x\neq 0,\|\hat{x}\|\geq\dfrac{\|\hat{x}(x)\|}{\|x\|}=\|x\|$.

Thus $\|\hat{x}\|=\|x\|$ for all x. Clearly $x\rightarrow\hat{x}$ is linear.

Further $(\hat{x}o\hat{y})(z)=\hat{x}(\hat{y}(z))=\hat{x}(z*y)$

$$=(z*y)*x=z*(y*x)=\left(\widehat{y*x}\right)z\Rightarrow\hat{x}\circ\hat{y}=\widehat{y*x}$$

$$\Rightarrow\|\hat{x}\circ\hat{y}\|=\left\|\widehat{y*x}\right\|=\|y*x\|\leq\|y\|\|x\|=\|\hat{y}\|\|\hat{x}\|=\|\hat{x}\|\|\hat{y}\|.$$

As a consequence we have $\hat{\hat{f}}:L_X\rightarrow L_{L_X}$ that satisfies:

(i) $\hat{f}\rightarrow\hat{\hat{f}}$ is linear,

(ii) $\widehat{\hat{f}\circ\hat{g}}=\widehat{\widehat{g\circ f}}$ and

(iii) $\left\|\hat{\hat{f}}\right\|=\left\|\hat{f}\right\|$ for $f,g\in L_X$.

To overcome the inconvinienice caused by (ii) we go a step further and define for $x\in X,\hat{\hat{x}}=\left(\widehat{\hat{x}}\right)$.

Now $x \to \hat{\hat{x}}$ maps X into L_{L_X} and satisfies:

(i) $x \to \hat{\hat{x}}$ is linear

(ii) $\widehat{\widehat{x \circ y}} = \widehat{\hat{x} \circ \hat{y}} = \hat{\hat{x}} \circ \hat{\hat{y}}$ and

(i) $\left\|\hat{\hat{x}}\right\| = \left\|\hat{x}\right\| = \|x\|$ for $x,y \in X$. This completes the proof.

Remark 4.3.
(i) Since L_{L_X} has an identity, the above theorem also shows that every normed near algebra can be embedded in a normed near algebra with identity.
(ii) If X is commutative, the embedding is obtained into L_X itself while L_{L_X} is needed when X is not commutative.

ACKNOWLEDGEMENT

The authors are thankful to Professor I. Ramabhadra Sarma for his valuable comments and suggestions.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

A MATHEMATICAL MODEL TO DIAGNOSE THE LEVEL OF DIABETES USING FUZZY LOGIC SYSTEM

By

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(Received : July 07, 2015; Revised in final form : September 20, 2015)

ABSTRACT

In the present work a simple and very effective mathematical model is designed for diagnosis of diabetes, specially of type 2. Sugar level, Triglyceride level, Blood pressure, Bodyweight are considered as input variables. Trapezoidal membership function is used for fuzzification process and defuzzification is done by *COG* technique. The fuzzy logic has been utilized in several different approaches to modeling the diagnostic process. This model addressed the level of type-2 diabetes on the basis of severity. The case of a diabetes patient is also discussed; in that case model diagnosed the level of severity with degree of precision 53.728 percent. These results are then also checked with the results obtained by scientific laboratory and found approximately correct.

2010 Mathematics Subject Classification: Primary 92B05; Secondary 03B52.

Keywords: Mathematical model, Diagnose the level of diabetes, Fuzzy logic system.

1. Introduction. The diagnosis of disease involves several levels of uncertainty and imprecision and it is inherent to medicine. A single disease may manifest itself quite differently, and with different intensities depending on patient. A single symptom may correspond to different diseases. On the other hand several diseases present in a patient may interact and interfere with usual description of any of diseases. Today the diabetes has become a major health problem among the people of all ages. This is one of the most popular diseases in the world. According to American diabetes association "Type 1 diabetes is usually diagnosed in

children and young adults, and was previously known as juvenile diabetes. Only 5% of people with diabetes have this form of the disease". Type 2 diabetes is the most common form of diabetes. Type 2 diabetes makes up about 90% of cases of diabetes with the other 10% due primarily to diabetes mellitus type 1 and gestational diabetes.

The World Health Organization (*WHO*) estimates that nearly 200 million people all over the world suffer from diabetes and this number is likely to be doubled by 2030 [10]. Even as nations prepare to mark World Diabetes Day on November 14, *WHO* says about 80% of the diabetes deaths occur in middle-income countries. In India, there are nearly 50 million diabetics, according to the statistics of the International Diabetes Federation [11]. Out of an estimated 62.4 million diabetics in India, 4 to 21 per cent women suffer from gestational diabetes mellitus (*GDM*) also called glucose intolerance or carbohydrate intolerance. There are an estimated 77.2 million people in India who are suffering from pre-diabetes[11]. These people are at high risk of getting diabetes. The Indian Council of Medical Research (*ICMR*) estimated that the country already had around 65.1 million diabetes patients. Only China, with 98.4 million cases, has more diabetes patients globally, according to V. Mohan, president of Madras Diabetes Research Foundation. **According to times of India-Health and fitness-5th Feb 2014 "India is home to nearly 62 million diabetics-second only to China which has over 92million diabetics.**

As the incidence of diabetes is on the rise there is a proportionate rise in the complications that are associated with diabetes. They point out that it is a very crucial stage and awareness on the part of people and administration about diabetes is very essential. People should be made aware and educated about their health and fitness level to reduce the number of patients in India. Most of the researchers have proposed different systems to diagnose diabetes. Fuzzy logic is also playing a great role in the diagnosis of diabetes. The Concept of a Linguistic Variable was given by Lofti A. Zadeh [1]. A Neural network and neuro-fuzzy systems has given for improving diabetes therapy [2], in which the use of a recurrent artificial neural network (*ANN*) is described which is able to predict *BGL* for a specific patient. This predicted *BGL* may then be used in a neuro-fuzzy expert system to offer short-term therapeutic advice regarding the

patient's diet, exercise and insulin regime (for insulin-dependent or Type 1 diabetics). Polat, K. and Gunes, S. used an expert system approach based on principal component analysis and adoptive neuro fuzzy inference system to diagnosis of diabetes [3]. This model is designed for diagnosis of Type -2 diabetes patients using if then rules. The process of fuzzfication is also introduced here.

Recently a work in this direction is to design A proposition for using mathematical models based on a fuzzy system with application [13], while by the same authors another recent work is to design a mathematical structure of fuzzy modeling of medical d diagnoses by using clustering models [14]. Further the derivation of interval type-2 fuzzy sets and systems on continuous domain: theory and application to heart diseases has also been worked by the same team [15].

2. Diabetes overview. We require energy, when we walk briskly, run for the bus, ride a bike, take an aerobic class and for our day to day chores, is provided by the glucose in blood. When food is taken, it is broken down to smaller components. Sugars and carbohydratesare thus broken down into glucose for the body to utilize them as energy source. The liver is also able to manufacture glucose. In a healthy person the hormone insulin, which is made by the beta cells of the pancreas, regulates how much glucose is in the blood. When there is excess glucose in the blood, insulin stimulates cells to absorb enough glucose from the blood for the energy that they need. Insulin also stimulates the liver to absorb and store any glucose that is excess in blood. Insulin release is triggered after a meal when there is rise in blood glucose. When blood glucose levels fall, during exercise for example, insulin levels fall too.

3. Diabetes. Diabetes describes a group of metabolic diseases in which the body cannot regulate the amount of sugar (glucose) in the blood. This is because the body does not produce enough insulin, produces no insulin, or has cells that do not respond properly to the insulin the pancreas produces. This results in too much glucose building up in the blood. This excess blood glucose eventually passes out of the body in urine. So, even though the blood has plenty of glucose, the cells are not getting it for their essential energy and growth requirements.

Types of diabetes. Diabetes type-1 **insulin-dependent diabetes (IDDM)**, or **early-onset diabetes**, Develops when the body does not produce insulin. People usually develop type 1 diabetes before their 40th year, often in early adulthood or teenage years. Patients with type 1 diabetes will need to take insulin injections for the rest of their life. Symptoms -Extreme thirst, frequent urination, Drowsiness or lethargy, Sudden weight loss, Sugar in the urine, Fruity odor on the breath, Heavy or labored breathing etc. **Gestational Diabetes** affects females during pregnancy. Women have very high levels of glucose in their blood. Diagnosis is made during pregnancy. Between10% to 20% of them will need to take some kind of blood-glucose-controlling medications. Undiagnosed or uncontrolled gestational diabetes can raise the risk of complications during childbirth. The baby may be bigger than he/she should be. Scientists from the National Institutes of Health and Harvard University found that women whose diets before becoming pregnant were high in animal fat and cholesterol had a higher risk for gestational diabetes

Diabetes Type-2 Adult-onset diabetes mellitus, or non-insulin-dependent-diabetes mellitus (*NIDDM*) develops when the body does not produce enough insulin for proper function, or the cells in the body do not react to insulin. The body does not produce enough insulin for proper function, or the cells in the body do not react to insulin (insulin resistance). Approximately 90% of all cases of diabetes worldwide are of this type. Some people may be able to control their type 2 diabetes symptoms by losing weight, following a healthy diet, doing plenty of exercise, and monitoring their blood glucose levels. However, type 2 diabetes is typically a progressive disease-it gradually gets worse - and the patient will probably end up have to take insulin, usually in tablet form. People usually develop type 2 diabetes after their 40th year.

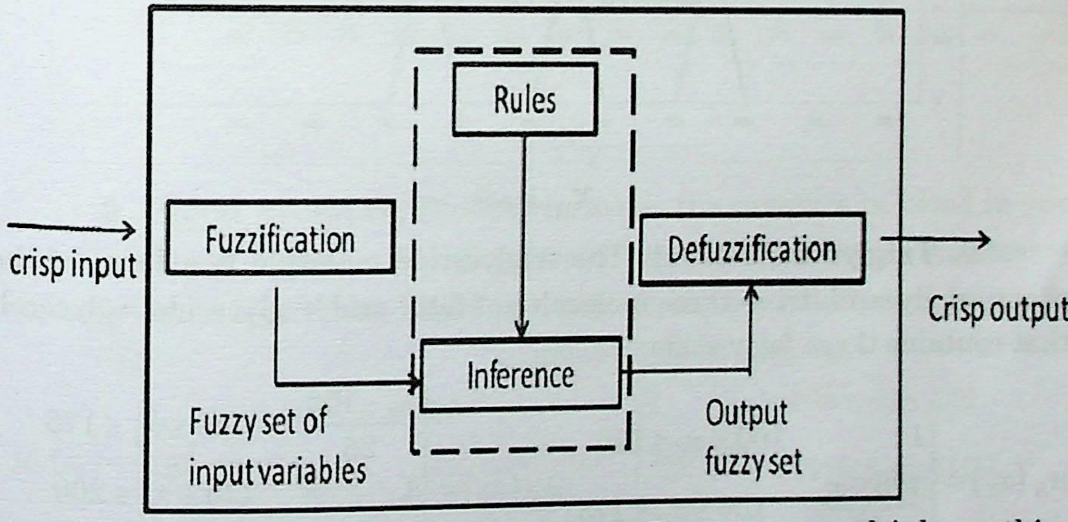
Symptoms of type 2 diabetes. Increased hunger, increased thirst, frequent urination, Blurred vision, Slow-healing sores or frequent infections.

Risk factors. High blood pressure, History of gestational diabetes, sedentary lifestyle, Age factor, polycystic ovary syndrome, Tri-glyceride level, Overweight

Complications of type 2 diabetes. Eye complications -diabetic retinopathy, Skin complications, Heart problems, Hypertension, Mental health, Hearing loss, Gumdisease, Neuropathy *HHNS* (Hyperosmolar, Hyperglycemic Nonketotic Syndrome), Erectidysfunction Nephropathy, Infections.

Fuzzy Logic. Fuzzy logic is an approach to computing based on "degrees of truth" rather than the usual "true or false" (1 or 0) Boolean logic on which the modern computer is based. The idea of fuzzy logic was first advanced by Dr. Lotfi Zadeh of the University of California at Berkeley in the 1960s. Dr. Zadeh was working on the problem of computer understanding of natural language. Natural language (like most other activities in life and indeed the universe) is not easily translated into the absolute terms of 0 and 1.Fuzzy logic includes 0 and 1 as extreme cases of truth (or "the state of matters" or "fact") but also includes the various states of truth in between so that, for example, the result of a comparison between two things could be not "tall" or "short" but ".38 of tallness."

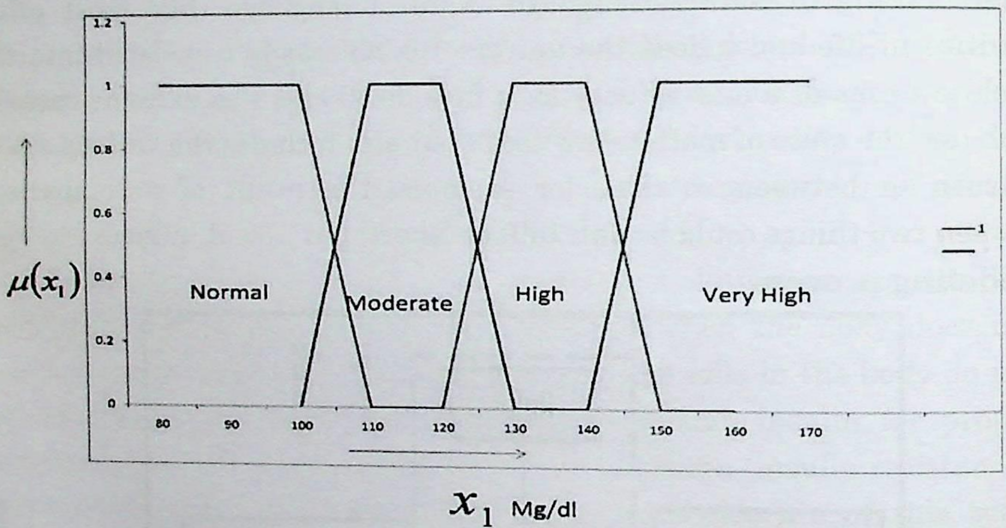
Modeling process



1. Impaired fasting glucose test. Do not eat or drink anything except water for 8-10 hours before a fasting blood glucose test.

$$\mu_N(x_1)=\begin{cases}1 & 80\le x_1\le 100\\ \dfrac{110-x_1}{10} & 100\le x_1\le 110\end{cases}\qquad \mu_M(x_1)=\begin{cases}\dfrac{x_1-100}{10} & 100\le x_1\le 110\\ 1 & 110\le x_1\le 120\\ \dfrac{110-x_1}{10} & 100\le x_1\le 110\end{cases}$$

$$\mu_H(x_1)=\begin{cases}\dfrac{x_1-120}{10} & 120\le x_1\le 130\\ 1 & 130\le x_1\le 140\\ \dfrac{150-x_1}{10} & 140\le x_1\le 150\end{cases}\qquad \mu_{V.H}(x_1)=\begin{cases}\dfrac{x_1-140}{10} & 140\le x_1\le 150\\ 1 & x_1\ge 150\end{cases}$$

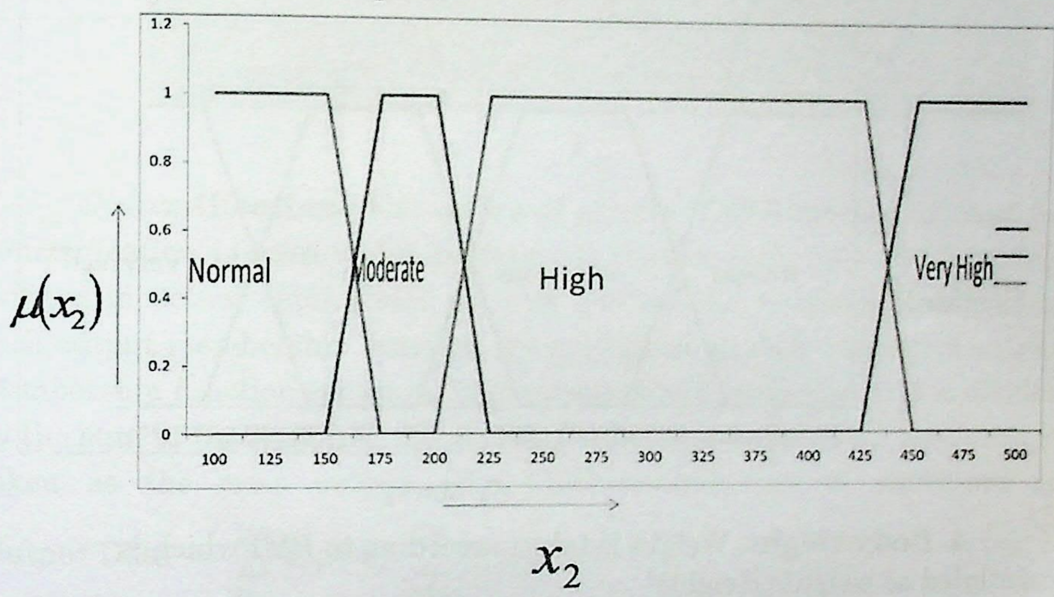


2. Triglyceride level. The triglyceride molecule is a form of the chemical glycerol (tri = three molecules of fatty acid + glyceride =glycerol) that contains three fatty acids

$$\mu_N(x_2)=\begin{cases}1 & 100\le x_2\le 150\\ \dfrac{175-x_2}{25} & 150\le x_2\le 175\end{cases}\qquad \mu_M(x_2)=\begin{cases}\dfrac{x_2-150}{25} & 150\le x_2\le 175\\ 1 & 175\le x_2\le 200\\ \dfrac{225-x_2}{25} & 200\le x_2\le 225\end{cases}$$

$$\mu_H(x_2)=\begin{cases}\dfrac{x_2-200}{25} & 200\le x_2\le 225\\ 1 & 225\le x_2\le 425\\ \dfrac{450-x_2}{25} & 425\le x_2\le 450\end{cases}$$

$$\mu_{V.H}(x_2)=\begin{cases}\dfrac{x_2-425}{25} & 425\le x_2\le 450\\ 1 & x_2\ge 450\end{cases}$$



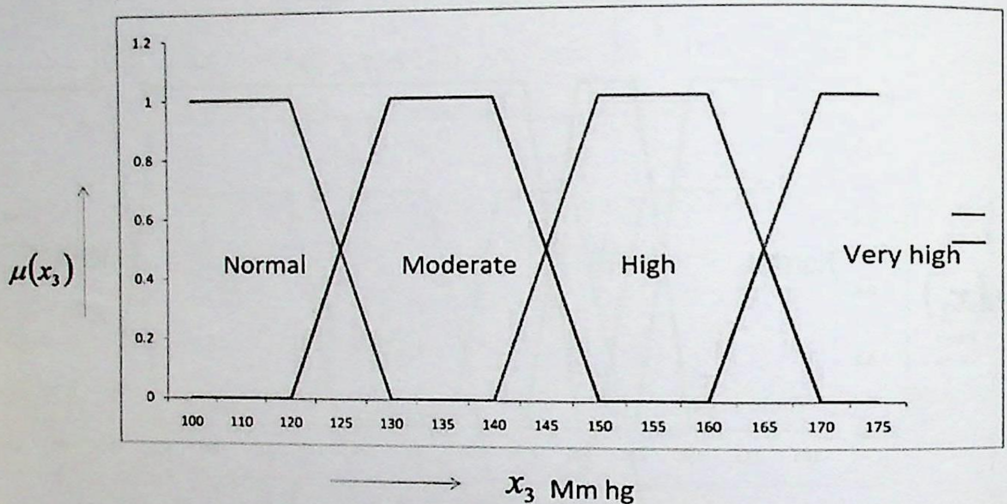
3. Blood pressure. Blood pressure is the pressure of blood in your arteries (blood vessels). Blood pressure is measured in millimeters of mercury (*mm Hg*). Your blood pressure is recorded as two figures. For example, 124/80 *mm Hg*.

$$\mu_N(x_3)=\begin{cases}1 & 100\le x_3\le 120\\ \dfrac{130-x_3}{10} & 120\le x_3\le 130\end{cases}\qquad \mu_N(x_3)=\begin{cases}1 & 100\le x_3\le 120\\ \dfrac{130-x_3}{10} & 120\le x_3\le 130\end{cases}$$

$$\mu_M(x_3)=\begin{cases}\dfrac{x_3-120}{10} & 120\le x_3\le 130\\ 1 & 130\le x_3\le 140\\ \dfrac{150-x_3}{25} & 140\le x_3\le 150\end{cases}$$

$$\mu_H(x_3)=\begin{cases}\dfrac{x_3-140}{10} & 140\le x_3\le 150\\ 1 & 150\le x_3\le 160\\ \dfrac{170-x_3}{10} & 160\le x_3\le 170\end{cases}$$

$$\mu_{V.H}(x_3)=\begin{cases}\dfrac{x_3-160}{25} & 160\le x_3\le 170\\ 1 & x_3\ge 170\end{cases}$$

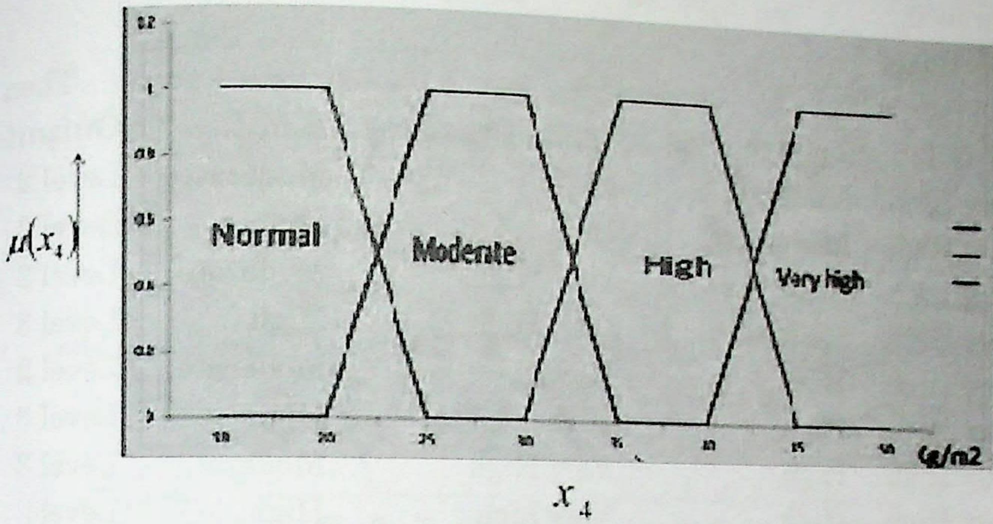


4. Bodyweight. Weight is taken according to *BMI* which is calculated as weight/(Height)^2

$$\mu_N(x_4)=\begin{cases}1 & 18\le x_4\le 20\\ \dfrac{25-x_4}{5} & 20\le x_4\le 25\end{cases} \quad \mu_N(x_4)=\begin{cases}1 & 18\le x_4\le 20\\ \dfrac{25-x_4}{5} & 20\le x_4\le 25\end{cases}$$

$$\mu_M(x_4)=\begin{cases}\dfrac{x_4-20}{5} & 20\le x_4\le 25\\ 1 & 25\le x_4\le 30\\ \dfrac{35-x_4}{5} & 30\le x_4\le 35\end{cases}$$

$$\mu_H(x_4)=\begin{cases}\dfrac{x_4-30}{5} & 30\le x_4\le 35\\ 1 & 35\le x_4\le 40\\ \dfrac{45-x_4}{5} & 40\le x_4\le 45\end{cases} \quad \mu_{V.H}(x_4)=\begin{cases}\dfrac{x_4-40}{25} & 40\le x_4\le 45\\ 1 & x_4\ge 45\end{cases}$$



Defuzzification. The centre of gravity (*COG*) method is used for defuzzification process which is the most popular technique and is widely utilized in actual applications. In this method the weighted strengths of each output membership function are multiplied by their respective output membership function center points and summed.Finally this area is divided by the sum of the weighted membership function strength and the result is taken as the crisp output. The *COG* method can be expressed as

$$\text{Output Data} = \sum_{i \in x_{\min}}^{x_{\max}} x_i . \mu(x_i) / \sum_{i \in x_{\min}}^{x_{\max}} \mu(x_i).$$

Case-Study

Mrs. M. Gupta Age- 62 years date: 13-02-2014

ABC Hospital, Noida India

Suffering with polydipsia, polyuria, irritation in urination swelling on feet, weakness.

Impaired glucose fasting test value (x1)-165mg/dl

Triglyceride test value(x_2) -212 mg/dl

Blood pressure (systolic), (x3)-165 mmHg, Weight (BMI), x_4-34.2kg/m2

The fuzzified value of the crisp inputs by the use of membership function defined for each fuzzy set for each linguistic variable is determined as follows

$\mu_N(x_1)=0$ $\mu_M(x_1) = 0$ $\mu_H(x_1) = 0,$ $\mu_{VH}(x_1) = 1$

$\mu_N(x_2)= 0,$ $\mu_M(x_2)= 0.52,$ $\mu_H(x_2) = 0.48,$ $\mu_{VH}(x_2) = 0,$

$\mu_N(x_3)= 0$ $\mu_M(x_3)= 0$ $\mu_H(x_3)= 0.5,$ $\mu_{VH}(x_3)=.5,$

$\mu_N(x_4) = 0,$ $\mu_M(x_4) = .16$ $\mu_H(x_4) = 0.84,$ $\mu_{VH}(x_4) = 0,$

Rule Base.

		If		Then
I.F.G.T.	**Triglyceride**	**Blood Pressure**	**B.M.I.**	**Output**
Very High	Moderate	High	Moderate	Level 2
Very High	Moderate	High	High	Level 2
Very High	Moderate	Very High	Moderate	Level 2
Very High	Moderate	Very High	High	Level 3
Very High	High	High	Moderate	Level 2
Very High	High	High	High	Level 3
Very High	High	Very High	Moderate	Level 3
Very High	High	Very high	High	Level4

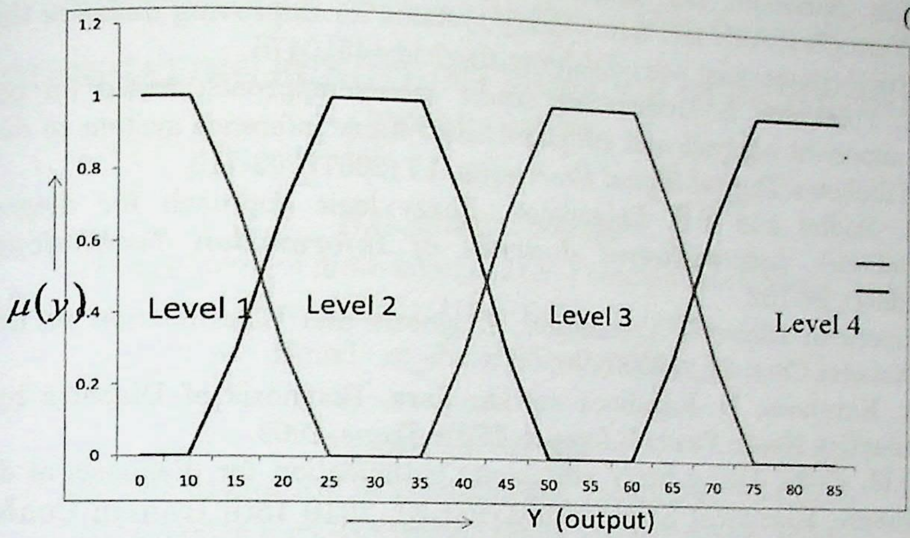
Execute the inference system. We use "Root Sum Square" (RSS) method to combine the effects of all applicable rules. Root sum square method scales the function at their respective magnitudes and computes the "fuzzy centroid" of the composite area. This method is more complicated mathematically than other methods level

$$Level2 = \sqrt{\sum_{j \in L_2} \left(\mu_{R_i}\right)^2} = \sqrt{(0.16)^2 + (0.5)^2 + (0.16)^2 + (0.16)^2} = 0.5766$$

$$Level\ 3 = \sqrt{\sum_{j \in L_3} \left(\mu_{R_i}\right)^2} = \sqrt{(0.5)^2 + (0.48)^2 + (0.16)^2} = 0.8430$$

$$Level\ 4 = \sqrt{\sum_{j \in L_4} \left(\mu_{R_i}\right)^2} = \sqrt{(0.48)^2} = 0.48$$

Output function.



Output of the decision of the expert system.

$$Output = \frac{0.5766 \times 0.30 + 0.8480 \times 0.55 + 0.48 \times 0.8}{0.5766 + 0.8430 + 0.48}$$
$$= 0.53728$$

This output shows that the patient is at level 3 diabetic with 53.728% degree of precision

Conclusion. Approximately 90% of all cases of diabetes worldwide are of this type. Some people may be able to control their type 2 diabetes symptoms by losing weight, following a healthy diet, doing plenty of exercise, and monitoring their blood glucose levels. However, type 2 diabetes is typically a progressive disease - it gradually gets worse - and the patient will probably end up have to take insulin, usually in tablet form. Once the level severity is diagnosed patient can control it in a better way. Here we obtained this fuzzy model provides good and exact information about the risk level of diabetes type 2.the performance of this fuzzy model is also shown here by taking a real data base of a patient. This model concludes that the patient is at the level 3 with degree of precision 53.728 percent.

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Jñānabha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

GENERATING FUNCTIONS THROUGH OPERATIONAL TECHNIQUES

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(*Received : December* 02, 2014; *Final form : March* 10, 2015)

ABSTRACT

The aim of the present paper is to illustrate the applications of differential operator $T_k = x(k + xD)$, $D \equiv \dfrac{d}{dx}$ to derive an interesting generating relation for generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12], [13]; also see Srivastava and Karlsson ([14,p.37, eqns (2.1) to (2.3)), with its special cases among others for Lauricella's $F_D^{(n)}$ [8], its confluent form $\phi_D^{(n)}$ and for hypergeometric function of three variables $F_3(x,y,z)$ of Srivastava [10].

2010 Mathematics Subject Classification : 33C70, 33C50

Keywords. Generating Functions, Operational techniques, Multiple hypergeometric functions, Generalized multiple hypergeometric function of Srivastava and Daoust.

1. Introduction. Garg [6] used the following operational formula due to Mittal [9] :

$$(1.1) \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} T^n_{\alpha+1+\beta n} f(x) = \frac{(1+v)^{\alpha+1}}{1-\beta v} f\left[x(1+v)\right] ,$$

where $v = xz(1+v)^{\beta+1}$, β being a constant, $f(x)$ admits a formal power series in x, $T_k = x(k + xD)$, $D \equiv \dfrac{d}{dx}$ and T_k^n means that the operator T_k is repeated n times, in obtaining a known generating relation due to Srivastava and Panda ([15],p.130, eqn. (4.3)) for multivariable H-function of Srivastava and Panda ([15],[16],[17], also see Srivastava, Gupta and Goyal [18]) obtained through quite different non-operational technique.

Our present paper shows the importance and utility of the differential operator T_k, in obtaining the generating relation for generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12],[13]; also see Srivastava and Karlsson [14,p.37, eqn. (2.1) to (2.3)). It also presents its interesting speical cases among others specially for Lauricella's $F_D^{(n)}$ [8], its confluent form $\phi_D^{(n)}$ and for triple hypergeometric series $F^{(3)}$ of Srivastava [10].

2. Main Generating Relation. In this Section, we derive the following generating relation for generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12],[13]) by operational technique:

$$(2.1)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!}(\alpha+\beta n)_n\, F^{A+1:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\,[\alpha+(\beta+1)n:\sigma_1,\ldots,\sigma_r]: \\ [(c):\psi',\ldots,\psi^{(r)}],[\alpha+\beta n:\sigma_1,\ldots,\sigma_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^r):\phi^r]\,; \\ [(d'):\delta'];\ldots;[(d^r):\delta^r]\,;\end{matrix}\; z_1,\ldots,z_r^{\sigma_r}\right),$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],[(b'):\phi'];\ldots;[b^{(r)}:\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}],[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\; z_1(1+v)^{\sigma_1},\ldots,z_r(1+v)^{\sigma_r}\right)$$

provided that $v = t(1+v)^{\beta+1}$, $|z_i|<1$, $\sigma_i>0$, and

$$1+\sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} - \sum_{j=1}^{A}\theta_j^{(i)} - \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} > 0, \quad i=1,\ldots,r.$$

Proof. In (1.1), choosing

(2.2) $$f(x)=F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(r)}\right],\left[(b'):\phi'\right];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right]; \\ \left[(c):\psi',\ldots,\psi^{(r)}\right],\left[(d'):\delta'\right];\ldots;\left[\left(d^{(r)}\right):\delta^{(r)}\right];\end{matrix}\; y_1x^{\sigma_1},\ldots y_rx^{\sigma_r}\right),$$

where $F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}$ is generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12],[13]) and $\sigma_i>0$, $i=1,\ldots,r$;

we have

(2.3) $$\frac{(1+v)^{\alpha}}{1-\beta v}F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(r)}\right],\left[(b'):\phi'\right];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right]; \\ \left[(c):\psi',\ldots,\psi^{(r)}\right],\left[(d'):\delta'\right];\ldots;\left[\left(d^{(r)}\right):\delta^{(r)}\right];\end{matrix}\right.$$

$$\left. y_1x^{\sigma_1}(1+v)^{\sigma_1},\ldots y_rx^{\sigma_r}(1+v)^{\sigma_r}\right)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}T^n_{\alpha+\beta n}\left\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(r)}\right]:\left[(b'):\phi'\right];\ldots;\left[(b^r):\phi^r\right]; \\ \left[(c):\psi',\ldots,\psi^{(r)}\right]:\left[(d'):\delta'\right];\ldots;\left[(d^r):\delta^r\right];\end{matrix}\right.\right.$$

$$\left. y_1x^{\sigma_1},\ldots y_rx^{\sigma_r}\right)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\ldots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^r)_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\ldots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^r)_{m_r\delta_j^{(r)}}}$$

$$y_1^{m_1}\cdots y_r^{m_r}T^n_{\alpha+\beta n}\left\{x^{m_1\sigma_1}+\ldots+x^{m_r\sigma_r}\right\},$$

where $v=xz(1+v)^{\beta+1}$.

Now making an appeal to (2.2) and

(2.4) $$T_k^n\left\{x^{\gamma}\right\}=(k+\gamma)_n\,x^{\gamma+n},$$

we obtain

(2.5) $$\frac{(1+v)^{\alpha}}{1-\beta v}F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(r)}\right],\left[(b'):\phi'\right];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right]; \\ \left[(c):\psi',\ldots,\psi^{(r)}\right],\left[(d'):\delta'\right];\ldots;\left[\left(d^{(r)}\right):\delta^{(r)}\right];\end{matrix}\right.$$

$$y_1 x^{\sigma_1}(1+v)^{\sigma_1},\dots,y_r x^{\sigma_r}(1+v)^{\sigma_r}\Bigg)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\dots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\dots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}}$$

$$\frac{(\alpha+\beta n)_n(\alpha+(\beta+1)n)_{m_1\sigma_1+\dots+m_r\sigma_r}}{(\alpha+\beta n)_{m_1\sigma_1+\dots+m_r\sigma_r}}y_1^{m_1}\dots y_r^{m_r}x^{n+m_1\sigma_1+\dots+m_r\sigma_r}$$

$$=\sum_{n=0}^{\infty}\frac{(zx)^n}{n!}(\alpha+\beta n)_n F^{A+1:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}],[\alpha+(\beta+1)n:\sigma_1,\dots,\sigma_r]:\\ [(c):\psi',\dots,\psi^{(r)}],[\alpha+\beta n:\sigma_1,\dots,\sigma_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^r):\phi^{(r)}];\\ [(d'):\delta'];\dots;[(d^r):\delta^{(r)}];\end{matrix}\ y_1x^{\sigma_1},\dots,y_rx^{\sigma_r}\right).$$

Now replacing z by t/x and y_i by z_i ($i=1,..,r$), we finally derive the main generating relation (2.1).

3. Special Case. When $\beta=0$. In this case $v=t(1+v)$. Therefore generating relation (2.1) reduces to

$$(3.1)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F^{A+1:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}],[\alpha+n:\sigma_1,\dots,\sigma_r]:\\ [(c):\psi',\dots,\psi^{(r)}],[\alpha:\sigma_1,\dots,\sigma_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\dots z_r\right),$$

$$=(1-t)^{-\alpha}F^{A:B';\dots;B^{(r)}}_{C:D';\dots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}],[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\dots,\psi^{(r)}],[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\right.$$

$$\left. z_1(1-t)^{-\sigma_1},\dots,z_r(1-t)^{-\sigma_r}\right),$$

provided that $\sigma_i > 0$, $|z_i| < 1$, $1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0$, $i=1,\ldots,r$.

4. Special Case. When $\beta=-1$. In this case $v=t$. Therefore generating relation (2.1) reduces to

$$(4.1)\quad \sum_{n=0}^{\infty}\binom{\alpha}{n}t^n F_{C+1:D';\ldots;D^{(r)}}^{A+1:B';\ldots;B^{(r)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(r)}\right],\left[\alpha+1:\sigma_1,\ldots,\sigma_r\right]:\\ \left[(c):\psi',\ldots,\psi^{(r)}\right],\left[\alpha+1-n:\sigma_1,\ldots,\sigma_r\right]:\end{matrix}\right.$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right];\\ \left[(d'):\delta'\right];\ldots;\left[\left(d^{(r)}\right):\delta^{(r)}\right];\end{matrix}\ z_1,\ldots z_r\right),$$

$$=(1+t)^{\alpha}F_{C:D';\ldots;D^{(r)}}^{A:B';\ldots;B^{(r)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(r)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{r}\right):\phi^{r}\right];\\ \left[(c):\psi',\ldots,\psi^{(r)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{r}\right):\delta^{r}\right];\end{matrix}\right.$$

$$\left.z_1(1+t)^{\sigma_1},\ldots,z_r(1+t)^{\sigma_r}\right),$$

valid if $|z_i|<1$, $\sigma_i>0$, and $1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0$, $i=1,\ldots,r$.

5 Special Cases. When $\sigma_1=\ldots=\sigma_r=1$ and specializing other parameters, (2.1) gives :

$$(5.1)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n F_D^{(r)}\left(\alpha+(\beta+1)n,b_1,\ldots,b_r;\alpha+\beta n;z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\prod_{i=1}^{r}\left[1-z_i(1+v)\right]^{-b_i},\quad |z_i|<1, i=1,\ldots,r$$

and

$$(5.2)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n \phi_D^{(r)}\left(\alpha+(\beta+1)n,b_1,\ldots,b_{r-1}-;\alpha+\beta n;z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\prod_{i=1}^{r-1}\left[1-z_i(1+v)\right]^{-b_i}\ e^{z_r(1+v)},\qquad |z_i|<1, i=1,\ldots,r.$$

(3.1) gives :

(5.3) $\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F_D^{(r)}(\alpha+n,b_1,...,b_r;\alpha;z_1,...,z_r)$

$$=(1-t)^{-\alpha}\prod_{i=1}^{r}\left[1-z_i/(1-t)\right]^{-b_i}, \qquad |z_i|<1, i=1,...,r$$

and

(5.4) $\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n \phi_D^{(r)}(\alpha+n,b_1,...,b_{r-1},-;\alpha;z_1,...,z_r)$

$$=(1-t)^{-\alpha}\prod_{i=1}^{r-1}\left[1-z_i/(1-t)\right]^{-b_i}.\ e^{z_r/(1-t)}, \qquad |z_i|<1, i=1,...,r;$$

while (4.1) gives

(5.5) $\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n F_D^{(r)}(\alpha+1,b_1,...,b_r;\alpha+1-n;z_1,...,z_r)$

$$=(1+t)^{\alpha}\prod_{i=1}^{r}\left[1-z_i(1+t)\right]^{-b_i} \qquad |z_i|<1, i=1,...,r$$

and

(5.6) $\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n \phi_D^{(r)}(\alpha+1,b_1,...,b_{r-1},-;\alpha+1-n;z_1,...,z_r)$

$$=(1+t)^{\alpha}\prod_{i=1}^{r}\left[1-z_i(1+t)\right]^{-b_i}, \qquad |z_i|<1, i=1,...,r;$$

where $v=t(1+v)^{\beta+1}$, $F_D^{(r)}$ is fourth multiple hypergeometric function due to Lauricella [8] while $\phi_D^{(r)}$ is its confluent form.

6. Other Special Cases.

Case (a) When $r=3$, $\sigma_1=\sigma_2=\sigma_3=1$ and specializing other paramters, the results (2.1), (3.1) and (4.1) reduce respectively to

(6.1) $\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+n\beta)_n F^{(3)}\left[\begin{matrix}(a),\alpha+(\beta+1)n:(b);(b');(b''):(c);(c');(c'');\\(f),\alpha+\beta n:(g);(g');(g''):(h);(h');(h'');\end{matrix} z_1,z_2,z_3\right]$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}F^{(3)}\left[\begin{matrix}(a):(b);(b');(b''):(c);(c');(c'');\\(f):(g);(g'):(g''):(h);(h');(h'');\end{matrix} z_1(1+v),z_2(1+v),z_3(1+v)\right],$$

where $v=t(1+v)^{\beta+1}$ and $F^{(3)}$ is generalized hypergeometric series of three variables due to Srivastva [10].

(6.2) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F^{(3)}\left[\begin{matrix}(a),\alpha+n:(b);(b');(b''):(c);(c');(c'');\\(f),\alpha:(g);(g');(g''):(h);(h');(h'');\end{matrix} z_1,z_2,z_3\right]$$

$$=(1-t)^{-\alpha}F^{(3)}\left(\begin{matrix}(a):(b);(b');(b''):(c);(c');(c'');\\(f):(g);(g'):(g''):(h);(h');(h'');\end{matrix}\frac{z_1}{(1-t)},\frac{z_2}{(1-t)},\frac{z_3}{(1-t)}\right).$$

(6.3) $$\sum_{n=0}^{\infty}t^n\binom{\alpha}{n}F^{(3)}\left(\begin{matrix}(a),\alpha+1:(b);(b');(b''):(c);(c');(c'');\\(f),\alpha+1-n:(g);(g');(g''):(h);(h');(h'');\end{matrix} z_1,z_2,z_3\right)$$

$$=(1+t)^{\alpha}F^{(3)}\left(\begin{matrix}(a):(b);(b');(b''):(c);(c');(c'');\\(f):(g);(g');(g''):(h);(h');(h'');\end{matrix} z_1(1+t),z_2(1+t),z_3(1+t)\right).$$

Case (b) When $r=2$, $\sigma_1=\sigma_2=1$ and specializing other parameters, we derive from (2.1)

(6.4) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n F_1\left(\alpha+(\beta+1)n,b_1,b_2;\alpha+\beta n;z_1,z_2\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\left[1-z_1(1+v)\right]^{-b_1}\left[1-z_2(1+v)\right]^{-b_2},\quad v=t(1+v)^{\beta+1},$$

where F_1 is Appell function of two variables.

(6.5) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n\,\phi_1\left(\alpha+(\beta+1)n,b_1;\alpha+\beta n;z_1,z_2\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\left[1-z_1(1+v)\right]^{-b_1}e^{z_2(1+v)},\quad v=t(1+v)^{\beta+1},$$

where ϕ_1 is confluent series of Appell's series F_1.

From (3.1), we derive

(6.6) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F_1(\alpha+n,b_1,b_2;\alpha;z_1,z_2)$$

$$=(1-t)^{-\alpha}\left[1-z_1/(1-t)\right]^{-b_1}\left[1-z_2/(1-t)\right]^{-b_2}$$

and

(6.7) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n\,\phi_1(\alpha+n,b_1;\alpha;z_1,z_2)=(1-t)^{-\alpha}\left[1-\frac{z_1}{1-t}\right]^{-b_1}e^{z_2/1-t}.$$

Further from (4.1), we deduce

$$(6.8)\quad \sum_{n=0}^{\infty}\binom{\alpha}{n} t^n F_1\left(\alpha+1,b_1,b_2;\alpha+1-n;z_1,z_2\right)$$
$$=(1+t)^{\alpha}\left[1-z_1(1+t)\right]^{-b_1}\left[1-z_2(1+t)\right]^{-b_2}.$$

and

$$(6.9)\quad \sum_{n=0}^{\infty}\binom{\alpha}{n} t^n \phi_1\left(\alpha+1,b_1;\alpha+1-n;z_1,z_2\right)$$
$$=(1+t)^{\alpha}\left[1-z_1(1+t)\right]^{-b_1} e^{z_2(1+t)}.$$

Case (c) When $r=1$ and $\sigma_1=1$ the generating relations (2.1), (3.1) and (4.1) reduce respectively to

$$(6.10)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n \;{}_1F_1\left[\begin{matrix}\alpha+(\beta+1)n;\\ \alpha+\beta n;\end{matrix}\, z\right]=\frac{(1+v)^{\alpha}}{1-\beta v}e^{(1+v)z},$$

where $v=t(1+v)^{\beta+1}$,

$$(6.11)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n \;{}_1F_1\left[\begin{matrix}\alpha+n;\\ \alpha;\end{matrix}\, z\right]=(1-t)^{-\alpha} e^{-z(1-t)}$$

and

$$(6.12)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}\binom{\alpha}{n} \;{}_1F_1\left[\begin{matrix}\alpha+1;\\ \alpha+1-n;\end{matrix}\, z\right]=(1+t)^{\alpha} e^{z(1+t)}.$$

7. Some more Generating Functions of other Multiple Hypergeometric Functions. In this section, we shall derive generating relations of multiple hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ with their confluent forms $\phi_2^{(r)}$, $\psi_2^{(r)}$ of Lauricella [8], ${}^{(k)}_{(1)}E_D^{(r)}$, ${}^{(k)}_{(2)}E_D^{(r)}$ of Exton ([4],[5]), ${}^{(k)}_{(1)}E_C^{(r)}$ of Chandel [1], ${}^{(k)}F_{AC}^{(r)}$, ${}^{(k)}F_{AD}^{(r)}$, ${}^{(k)}F_{BD}^{(r)}$ and their confluent forms ${}^{(k)}_{(1)}\phi_{AC}^{(r)}$, ${}^{(k)}_{(2)}\phi_{AC}^{(r)}$, ${}^{(k)}_{(1)}\phi_{AD}^{(r)}$, ${}^{(k)}_{(1)}\phi_{BD}^{(r)}$, ${}^{(k)}_{(2)}\phi_{BD}^{(r)}$ of Chandel and Gupta [2], ${}^{(k)}F_{CD}^{(r)}$ of Karlsson [7] and for its confluent forms ${}^{(k)}_{(1)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(2)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(3)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(4)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(5)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(6)}\phi_{CD}^{(r)}$ due to Chandel and Vishwakarma [3] and also for more confluent forms ${}^{(k)}_{(2)}\phi_{AD}^{(r)}$, ${}^{(k)}_{(3)}\phi_{BD}^{(r)}$, ${}^{(k)}_{(1)}\phi_{D}^{(r)}$, ${}^{(k)}_{(2)}\phi_{D}^{(r)}$, ${}^{(k)}_{(1)}\phi_{C}^{(r)}$ due to Vishwakarma [19].

Choosing $\sigma_1 = \ldots = \sigma_r = 1$ and specializing other parameters in (2.1), we derive the following generating relations :

$$(7.1)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+n\beta)_n\, F^{2:1,\ldots;1}_{1:1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:[b_1:1];\ldots;[b_r:1];\\ [\alpha+n\beta:1,\ldots,1]:[c_1:1];\ldots;[c_r:1]\,;\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}F_A^{(r)}\left(a,b_1,\ldots,b_r;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$\sum_{i=1}^{r}\left|(1+v)z_i\right|<1\ ,$$

$$(7.2)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+n\beta)_n\, F^{1:2,\ldots;2}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1]:[a_1:1],[b_1:1];\ldots;[a_r1],[b_r:1];\\ [\alpha+n\beta:1,\ldots,1]:[c:1,\ldots,1]:-;\ldots;-;\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}F_B^{(r)}\left(a_1,\ldots,a_r,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$\left|(1+v)z_i\right|<1,\ \ i=1,\ldots,r\,,$$

$$(7.3)\quad \sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{3:-;\ldots;-}_{1:1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],[b:1,\ldots,1]:-;\ldots;-;\\ [\alpha+n\beta:1,\ldots,1]:[c_1:1];\ldots;[c_r:1];\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}F_C^{(r)}\left(a,b;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$\sum_{i=1}^{r}\left|\left[(1+v)z_i\right]^{1/2}\right|<1\ ,$$

$$(7.4)\quad \sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{2:-;\ldots;-}_{1:1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:-;\ldots;-;\\ [\alpha+n\beta:1,\ldots,1]:[c_1:1];\ldots;[c_r:1];\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\psi_2^{(r)}\left(a;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.5)\quad \sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{1:1,\ldots;1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1]:[b_1:1];\ldots;[b_r:1];\\ [\alpha+n\beta:1,\ldots,1],[c:1,\ldots,1]:-;\ldots;-;\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\phi_2^{(r)}\left(b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.6)\ \sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{2:1,\dots;1}_{3:-;\dots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\dots,1],[a:1,\dots,1]: \\ [\alpha+n\beta:1,\dots,1]:[c:1,\dots,1,-,\dots,-],[c':-,\dots,-,1,\dots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\dots;[b_r:1]; \\ -;\dots;-;\end{matrix} z_1,\dots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}E^{(r)}_{D}\left(a,b_1,\dots,b_r;c,c';z_1(1+v),\dots,z_r(1+v)\right),$$

$$(7.7)\ \sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{3:1,\dots;1}_{2:-;\dots;-}$$

$$\left(\begin{matrix}[\alpha+(\beta+1)n:1,\dots,1],[a:1,\dots,1,-,\dots,-],[a':-,\dots,-,1,\dots,1]: & b_1,\dots b_r; \\ [\alpha+n\beta:1,\dots,1]:[c:1,\dots,1]: & -;\dots;-;\end{matrix} z_1,\dots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}E^{(r)}_{D}\left(a,a',b_1,\dots,b_r;c;z_1(1+v),\dots,z_r(1+v)\right).$$

$$(7.8)\ \sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{4:-;\dots;-}_{1:1;\dots;1}$$

$$\left(\begin{matrix}[\alpha+(\beta+1)n:1,\dots,1],[a:1,\dots,1,-,\dots,-],[a':-,\dots,-,1,\dots,1],[b:1,\dots,1]:-;\dots;-; \\ [\alpha+n\beta:1,\dots,1]:\end{matrix} z_1,\dots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}E^{(r)}_{C}\left(a,a',b;c_1,\dots,c_r;z_1(1+v),\dots,z_r(1+v)\right),$$

where ${}^{(k)}_{(1)}E^{(r)}_{D}$, ${}^{(k)}_{(2)}E^{(r)}_{D}$ are multiple hypergeometric functions of Exton ([4],[5]) related to Lauricella's $F^{(r)}_{D}$ while ${}^{(k)}_{(1)}E^{(r)}_{C}$ is multiple hypergeometric function of Chandel [1] related to Lauricella's $F^{(n)}_{C}$.

$$(7.9)\ \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n t^n}{n!}F^{3:-;\dots;-;1,\dots,1}_{1:1,\dots,1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\dots,1],[a:1,\dots,1], \\ [\alpha+\beta n:1,\dots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b:1,\dots,1,-,\dots,-,]:-;\dots;-;[b_{k+1}:1];\dots;[b_r:1]; \\ [c_1:1];\dots;[c_r:1];\end{matrix} z_1,\dots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}F^{(r)}_{AC}\left(a,b,b_{k+1},\dots,b_r;c_1,\dots,c_r;z_1(1+v),\dots,z_r(1+v)\right),$$

$$(7.10)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n\, t^n}{n!} F^{2:1;\ldots;1}_{2:-;\ldots;-;1,\ldots,1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]: \\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-,\ldots,-]:\end{array}\right.$$

$$\left.\begin{array}{l}[b_1:1];\ldots;[b_r:1]; \\ -;\ldots;-;[c_{k+1}:1];\ldots;[c_r:1];\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}F^{(r)}_{AD}\left(a,b_1,\ldots,b_r;c,c_{k+1},\ldots,c_n;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.11)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n\, t^n}{n!} F^{2:1;\ldots;1;2;\ldots;2}_{2:-;\ldots;-}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1,-,\ldots,-]: \\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1]:\end{array}\right.$$

$$\left.\begin{array}{l}[b_1:1];\ldots;[b_k:1];[a_{k+1}:1],[b_{k+1}:1];\ldots;[a_n:1],[b_n:1]; \\ -;\ldots;-;\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}F^{(r)}_{BD}\left(a,a_{k+1},\ldots,a_r,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.12)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n\, t^n}{n!} F^{3:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-;1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],[b:-;\ldots;-;1,\ldots,1]: \\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:\end{array}\right.$$

$$\left.\begin{array}{l}[b_1:1];\ldots;[b_k:1];-;\ldots;-; \\ -;\ldots;-;[c_{k+1}:1];\ldots;[c_r:1]\,;\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}F^{(r)}_{CD}\left(a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.13)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!} t^n F^{3:-;\ldots;-}_{1:1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],[b:1,\ldots,1,-,\ldots,-]:-;\ldots;-; \\ [\alpha+\beta n:1,\ldots,1],[c_1:1];\ldots;[c_r:1];\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi^{(r)}_{AC}\left(a,b;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.14)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!} t^n F^{2:-;\ldots;-;1;\ldots;1}_{1:1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],-;\ldots;-;[b_{k+1}:1];\ldots;[b_r:1]; \\ [\alpha+\beta n:1,\ldots,1],[c_1:1];\ldots;[c_r:1];\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}\phi^{(r)}_{AC}\left(a,b_{k+1},\ldots,b_r;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.15)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:[b_1:1];\ldots;[b_r:1]\ ;\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:-;\ldots;-;\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi^{(r)}_{AD}\left(a,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right)$$

$$(7.16)\quad \frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1,\ldots,1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:[b_1:1];\ldots;[b_r:1]\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:-;\ldots;-;\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi^{(r)}_{BD}\left(a,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.17)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{1:1;\ldots;1;2;\ldots;2}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha(\beta+1)n:1,\ldots,1]:[b_1:1];\ldots;[b_k:1]\ ;\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[a_{k+1}:1],[b_{k+1}:1];\ldots;[a_r:1],[b_r:1];\\ -;\ldots;-;\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}\phi^{(r)}_{BD}\left(a_{k+1},\ldots,a_r,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.18)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{3:-;\ldots;-}_{2:-;\ldots;-;1,\ldots,1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1]:[a:1,\ldots,1],[b:-;\ldots;-,1,\ldots,1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:-;\ldots;-;\end{matrix}\right.$$

$$\left.\begin{matrix}-;\ldots;-;\\ [c_{k+1}:1];\ldots;[c_r:1];\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi^{(r)}_{CD}\left(a,b;c,c_{k+1},\ldots,c_n;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(7.19)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-1,\ldots,-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];-;\ldots;-;\\ -;\ldots;-;[b_{k+1}:1];\ldots;[b_r:1];\end{matrix}\ z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v}\, {}^{(k)}_{(2)}\phi^{(r)}_{CD}\left(a, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v_r)\right),$$

$$(7.20)\quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-;1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[b:-,\ldots,-,1,\ldots,1]: \\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-,\ldots,-]:\end{array}\right.$$

$$\left.\begin{array}{l}[b_1:1];\ldots;[b_k:1];-;\ldots;-; \\ -;\ldots;-;[c_{k+1}:1];\ldots;[c_r:1];\end{array} z_1, \ldots, z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v}\, {}^{(k)}_{(3)}\phi^{(r)}_{CD}\left(b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)\right),$$

$$(7.21)\quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{3:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots1]: \\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-,\ldots,-]:\end{array}\right.$$

$$\left.\begin{array}{l}[b:-,\ldots,-,1,\ldots,1]:[b_1:1];\ldots;[b_k:1];-;\ldots;- ; \\ -;\ldots;-;\end{array} z_1, \ldots, z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v}\, {}^{(k)}_{(4)}\phi^{(r)}_{CD}\left(a, b, b_1, \ldots, b_k; c; z_1(1+v), \ldots, z_r(1+v)\right),$$

$$(7.22)\quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{3:1;\ldots;1;-;\ldots;-}_{1:-;\ldots;-;1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots1]: \\ [\alpha+\beta n:1,\ldots,1]:-;\ldots;-;\end{array}\right.$$

$$\left.\begin{array}{l}[b:-,\ldots,-,1,\ldots,1]:[b_1:1];\ldots;[b_k:1];-;\ldots;- ; \\ [c_{k+1}:1];\ldots;[c_r:1]\ ;\end{array} z_1, \ldots, z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v}\, {}^{(k)}_{(5)}\phi^{(r)}_{CD}\left(a, b, b_1, \ldots, b_k; c_{k+1}, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)\right),$$

$$(7.23)\quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1;-;\ldots;-}_{1:-;\ldots;-;1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots1]: \\ [\alpha+\beta n:1,\ldots,1]:-;\ldots;-;\end{array}\right.$$

$$\left.\begin{array}{l}[b_1:1];\ldots;[b_k:1];-;\ldots;- ; \\ [c_{k+1}:1];\ldots;[c_r:1]\end{array} z_1, \ldots, z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} {}^{(k)}_{(6)}\phi^{(r)}_{CD}\left(a, b_1, \ldots, b_k; c_{k+1}, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)\right),$$

(7.24) $$\sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1}_{1:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1]: \\ [\alpha+\beta n:1,\ldots,1]:-;\ldots;-;\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1]; \\ [c_{k+1}:1];\ldots;[c_r:1];\end{matrix} z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} {}^{(k)}_{(2)}\phi^{(r)}_{AD}\left(a, b_1, \ldots, b_r; c_{k+1}, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)\right),$$

(7.25) $$\sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1;2;\ldots;2}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1,-;\ldots;-]: \\ [\alpha+\beta n:1,\ldots,1]:[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];[a_{k+1}:1],[b_{k+1}:1];\ldots;[a_r:1],[b_r:1]; \\ -;\ldots;-;\end{matrix} z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} {}^{(k)}_{(3)}\phi^{(r)}_{BD}\left(a, a_{k+1}, \ldots, a_r, b_1, \ldots, b_k; c; z_1(1+v), \ldots, z_r(1+v)\right),$$

(7.26) $$\sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1]: \\ [\alpha+\beta n:1,\ldots,1]:[c:1,\ldots,1,-;\ldots;-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1]; \\ -;\ldots;-;\end{matrix} z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} {}^{(k)}_{(1)}\phi^{(r)}_{D}\left(a, b_1, \ldots, b_r; c; z_1(1+v), \ldots, z_r(1+v)\right),$$

(7.27) $$\sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1,-,\ldots,-]: \\ [\alpha+\beta n:1,\ldots,1]:[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1]; \\ -;\ldots;-;\end{matrix} z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} {}^{(k)}_{(2)}\phi_D^{(r)}\left(a, b_1, \ldots, b_r; c; z_1(1+v), \ldots, z_r(1+v)\right),$$

$$(7.28) \quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta n)_n}{n!} t^n F^{3:-;\ldots;-}_{1:1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1,-;\ldots;-]: \\ [\alpha+\beta n:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b:1,\ldots,1]:-;\ldots;-; \\ [c_1:1];\ldots;[c_r:1];\end{matrix} z_1, \ldots, z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} {}^{(k)}_{(1)}\phi_C^{(r)}\left(a, b; c_1, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)\right).$$

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Jñānabha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

APPLICATIONS OF DIFFERENCE OPERATORS IN TRANSFORMATIONS OF CERTAIN MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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(Received : January 07, 2015)

ABSTRACT

In the present paper, making an appeal to difference operator, we derive many transformations of certain multiple hypergeometric functions of several variables introduced and studied by Chandel and Gupta ([6],[7]). We also discuss their special cases. In last sections, we also derive the transformations of certain quadruple hypergeometric functions due to Exton ([10,[11]) and Sharma-Parihar [19].

2010 Mathematics Subject Classification : Primary 33C67, 33C65, Secondary 39A70

Keywords. Difference operators, Transformations, Multiple hypergeometric functions, Quadruple hypergeometric functions

1. Introduction. Agrawal ([1],[2]) and Gupta and Agrawal ([12],[13],[14]) employed difference operators E and Δ to obtain various transformations of hypergeometric series. Further Gupta and Agrawal [15] established a transformation of Louricella's multiple hypergeometric function $F_A^{(n)}$ ([16], also see [18]). Recently, making an appeal to difference operators Δ and E, Chandel [8], obtained various transfermation formulae for some multiple hypergeometric functions of several variables introduced and studied by Chandel et al. ([4],[5] and discussed their special cases.

Here motivated by above work, in the present paper, we derive various transformations for some of our multiple hypergeometric functions ${}^{(k,k')}F_{AC}^{(n)}$, ${}^{(k,k')}F_{AD}^{(n)}$ and their influent forms ${}^{(k,k')}_{(1)}\phi_{AC}^{(n)}$, ${}^{(k,k')}_{(2)}\phi_{AC}^{(n)}$, ${}^{(k,k')}_{(2)}\phi_{AD}^{(n)}$, ${}^{(k,k')}_{(3)}\phi_{AD}^{(n)}$ related to Lauricells functions introduced and studied by Chandel and

Gupta ([6],[7]). Some interesting special cases are also discussed. Our results may be useful in various related topics of mathematical analysis.

2. Formulae Required. In this section, we write following wellknown results, which will be used in our further investigations:

Lemma. (Multinomial Theorem) :

(2.1) $\left(x_1+\ldots+x_n\right)^k=\sum_{\sum m_i=k}\frac{k!\,x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_n!}$,

Lemma. (Srivatava [17, p.4])

(2.2) $\sum_{M=0}^{\infty}\frac{(a)_M}{M!}\left(x_1+\ldots+x_n\right)^M=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}(a)_{m_1+\ldots+m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$,

(2.3) $E^m\frac{\Gamma(b)}{\Gamma(c)}=\frac{\Gamma(b+m)}{\Gamma(c+m)}$,

(2.4) $\Delta^m\frac{\Gamma(b)}{\Gamma(c)}=(-1)^m\frac{\Gamma(b)}{\Gamma(c)}\frac{(c-b)_m}{(c)_m}$.

3.Transformations of Multiple Hypergeometric Functions. Consider

$$A=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}\left[1-\left(x_1E_1+\ldots+x_nE_n\right)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}$$

$$=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}\,x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_n!}E_1^{m_1}\ldots E_n^{m_n}$$

$$\left\{\frac{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}.$$

Now making an appeal to (2.3), we derive

(3.1) $A=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b_1+\ldots+b_k)_{m_1+\ldots+m_k}(b_{k+1}+\ldots+b_{k'})_{m_{k+1}+\ldots+m_{k'}}}{(c_1)_{m_1}\ldots(c_n)_{m_n}}$

$$\left(b_{k'+1}\right)_{m_{k'+1}}\cdots\left(b_n\right)_{m_n}\frac{x_1^{n_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}F_{AC}^{(n)}\left(a,b_1+\ldots+b_k,b_{k+1}+\ldots+b_{k'},b_{k'+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right).$$

Also, we can write

$$A=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}\left[1-(x_{k'+1}+\ldots+x_n)\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\ldots+x_{k'}E_{k'}+x_{k'+1}\Delta_{k'+1}+\ldots+x_n\Delta_n}{1-(x_{k'+1}+\ldots+x_n)}\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}.$$

Now making an appeal to (2.2), we have

$$A=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}\left[1-(x_{k'+1}+\ldots+x_n)\right]^{-a}$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}(a)_{m_1+\ldots+m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{E_1^{m_1}\ldots E_{k_1}^{m_{k'}}\Delta_{k'+1}^{m_{k'+1}}\ldots\Delta_n^{m_n}}{\left[1-(x_{k'+1}+\ldots+x_n)\right]^{m_1+\ldots+m_n}}$$

$$\left\{\frac{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}.$$

Further employing (2.3) and (2.4), we obtain

(3.2) $$A=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_1+\ldots+b_k)\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}\left[1-(x_{k'+1}+\ldots+x_n)\right]$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}}{\left[1-(x_{k'+1}+\ldots+x_n)\right]^{m_1+\ldots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$\frac{\Gamma(b_1+\ldots+b_k+m_1+\ldots+m_k)\Gamma(b_{k+1}+\ldots+b_{k'}+m_{k+1}+\ldots+m_{k'})}{\Gamma(c_1+m_1)\ldots\Gamma(c_{k'}+m_{k'})}$$

$$\frac{(-1)^{m_{k'+1}+\ldots+m_n}\Gamma(b_{k'+1})\ldots\Gamma(b_n)}{\Gamma(c_{k'+1})\ldots\Gamma(c_n)}\frac{(c_{k'+1}-b_{k'+1})_{m_{k'+1}}\ldots(c_n-b_n)_{m_n}}{(c_{k'+1})_{m_{k'+1}}\ldots(c_n)_{m_n}}$$

$$=\left[1-(x_{k'+1}+\ldots+x_n)\right]^{-a}$$

$${}^{(k,k)}F_{AC}^{(n)}\left(a,b_1+\ldots+b_k,b_{k+1}+b_{k'},c_{k'+1}-b_{k'+1},\ldots,c_n-b_n;c_1,\ldots,c_n;\right.$$

$$\frac{x_1}{1-(x_{k'+1}+...+x_n)},...,\frac{x_{k'}}{1-(x_{k'+1}+...+x_n)},\frac{-x_{k'+1}}{1-(x_{k'+1}+...+x_n)},...,\frac{-x_n}{1-(x_{k'+1}+...+x_n)}\Bigg).$$

Hence equating the both values of A from (3.1) and (3.2), we establish the transformation

(3.3) $${}^{(k,k')}F_{AC}^{(n)}(a,b,b',b_{k'+1},...,b_n;c_1,...,c_n;x_1,...,x_n)$$

$$=\left[1-(x_{k'+1}+...+x_n)\right]^{-a'}\,{}^{(k,k')}F_{Ac}^{(n)}\Big(a,b,b',c_{k'+1}-b_{k'+1},...,c_n-b_n,c_1,...,c_n;$$

$$\frac{x_1}{1-(x_{k'+1}+...+x_n)},...,\frac{x_{k'}}{1-(x_{k'+1}+...+x_n)},\frac{-x_{k'+1}}{1-(x_{k'+1}+...+x_n)},...,\frac{-x_n}{1-[x_{k'+1}+...+x_n]}\Bigg).$$

$$1\le k\le k'<n;k,k',n\in N.$$

If $r_1,...,r_n$ are associated radii of convergence the series ${}^{(k,k')}F_{AC}^{(n)}$, then

$$\left(\frac{1}{\sqrt{r_1}}+...+\frac{1}{\sqrt{r_k}}\right)^2+\left(\frac{1}{\sqrt{r_{k+1}}}+...+\frac{1}{\sqrt{r_{k'}}}\right)^2+\frac{1}{r_{k'+1}}+...+\frac{1}{r_n}=1,$$

which suggests $n-k'-1$ results in the folloing unified form:

(3.4) $${}^{(k,k')}F_{AC}^{(n)}(a,b,b',b_{k'+1},...,b_n;c_1,...,c_n;x_1,...,x_n)$$

$$=\left[1-(x_{r+1}+...+x_n)\right]^{-a}\,{}^{(k,k')}F_{AC}^{(n)}\Big(a,b,b',b_{k'+1},...,b_r,c_{r+1}-b_{r+1},...,c_n-b_n;c_1,...,c_n$$

$$\frac{x_1}{1-(x_{r+1}+...+x_n)},...,\frac{x_r}{1-(x_{r+1}+...+x_n)},\frac{-x_{r+1}}{1-(x_{r+1}+...+x_n)},...,\frac{-x_n}{1-(x_{r+1}+...+x_n)}\Bigg),$$

$$r=k',...,n-1,\ 1\le k\le k'\le r<n;\ k,k',r,n\in N.$$

Special Cases .

(I) For $r=k'$ (3.4) reduces to (3.3).

(II) For $k'=k$ (3.3) reduces to the result due to Chandel ([8],p. (2.2))

Further considering

$$B=\frac{\Gamma(c_1+...+c_k)\Gamma(c_{k+1}+...+c_{k'})\Gamma(c_{k'+1})...\Gamma(c_n)}{\Gamma(b_1)...\Gamma(b_n)}\left[1-(x_1E_1+...+x_nE_n)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1)...\Gamma(b_n)}{\Gamma(c_1+...+c_k)\Gamma(c_{k+1}+...+c_{k'})\Gamma(c_{k'+1})...\Gamma(c_n)}\right\}$$

and making an appeal to (2.2) and (2.3), we obtain

(3.5) $$B={}^{(k,k')}F_{AD}^{(n)}(a,b_1,...,b_n;c_1+...+c_k,c_{k+1}+...+c_{k'},c_{k'+1},...,c_n;x_1,...,x_n)\ .$$

Also, we can write in another way

$$B=\frac{\Gamma(c_1+...+c_k)\Gamma(c_{k+1}+...+c_{k'})\Gamma(c_{k'+1})...\Gamma(c_n)}{\Gamma(b_1)...\Gamma(b_n)}\left[1-(x_{k'+1}+...+x_n)\right]^{-a}$$

$$\left[1-\frac{x_1E_1+...+x_{k'}E_{k'}+x_{k'+1}\Delta_{k'+1}+...+x_n\Delta_n}{1-(x_{k'+1}+...+x_n)}\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1)...\Gamma(b_n)}{\Gamma(c_1+...+c_k)\Gamma(c_{k+1}+...+c_{k'})\Gamma(c_{k'+1})...\Gamma(c_n)}\right\}.$$

Now making an appeal to (2.2), (2.3) and (2.4), we establish

(3.6) $$B=\left[1-(x_{k'+1}+...+x_n)\right]^{-a}\ {}^{(k,k')}F_{AD}^{(n)}\Big(a,b_1,...,b_{k'},c_{k'+1}-b_{k'+1},...,c_n-b_n;$$
$$c_1+...+c_k,c_{k+1}+...+c_{k'},c_{k'+1},...,c_n;\frac{x_1}{1-(x_{k'+1}+...+x_n)},...,$$
$$\frac{x_{k'}}{1-(x_{k'+1}+...+x_n)},\frac{-x_{k'+1}}{1-(x_{k'+1}+...+x_n)},...,\frac{-x_n}{1-(x_{k'+1}+...+x_n)}\Big).$$

Therefore, equating both values of B from (3.5) and (3.6) we finally derive the transformation

(3.7) $${}^{(k,k')}F_{AD}^{(n)}(a,b_1,...,b_n;c,c',c_{k'+1},...,c_n;x_1,...,x_n)$$
$$=\left[1-(x_{k'+1}+...+x_n)\right]^{-a}\ {}^{(k,k')}F_{AD}^{(n)}\Big(a,b_1,...,b_{k'},c_{k'+1}-b_{k'+1},...,c_n-b_n;$$
$$c,c',c_{k'+1},...,c_n;\frac{x_1}{1-(x_{k'+1}+...+x_n)},...,\frac{x_{k'}}{1-(x_{k'+1}+...+x_n)},$$
$$\frac{-x_{k'+1}}{1-(x_{k'+1}+...+x_n)},...,\frac{-x_n}{1-(x_{k'+1}+...+x_n)}\Big),$$
$$1\le k\le k'<n;\ \ k,k',n\in N;$$

if $r_1,...,r_n$ are associated radii of convergence of the series ${}^{(k,k')}F_{AD}^{(n)}$ then $r_k+r_{k'}+r_{k'+1}+...+r_n=1$.

Special Case. For $k'=k$ it reduces to the result due to Chandel ([8],p. (2.11)

Making an appeal to the same technique the above result (3.7) can be generalized in the following unified from of $n-k'-1$ results :

(3.8) $${}^{(k,k')}F_{AD}^{(n)}(a,b_1,...,b_n;c,c',c_{k'+1},...,c_n;x_1,...,x_n)$$
$$=\left[1-(x_{r+1}+...+x_n)\right]^{-a}\ {}^{(k,k')}F_{AD}^{(n)}\big(a,b_1,...,b_r,c_{r+1}-b_{r+1},...,c_n-b_n;$$

$$c,c',c_{k'+1},\ldots,c_n;\frac{x_1}{1-(x_{r+1}+\ldots+x_n)},\ldots,\frac{x_r}{1-(x_{r+1}+\ldots+x_n)},$$

$$\left.\frac{-x_{r+1}}{1-(x_{r+1}+\ldots x_n)},\ldots,\frac{-x_n}{1-(x_{r+1}+\ldots+x_n)}\right),$$

$$1\le k\le k'\le r<n;\ k,k',n\in N, N=\{1,2,3,\ldots\}.$$

Special Cases. For $r=k'$, (3.8) reduces to (3.7).

For $k'=k$, (3.8) reduces to Chandel ([8],p., (3.1))

4. Transformations of Confluent Multiple Hypergeometric Functions. Considering

$$C=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1}+\ldots+b_n)}\left[1-(x_1E_1+\ldots+x_nE_n)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1}+\ldots+b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}$$

$$\left\{\frac{\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1}+\ldots+b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\},$$

and applying (2.2), (2.3), we derive

(4.1) $C={}^{(k,k')}_{(1)}\phi^{(n)}_{AC}(a,b_{k+1}+\ldots+b_{k'},b_{k'+1}+\ldots+b_n;c_1,\ldots,c_n,x_1,\ldots,x_n)$.

Now again writing

$$C=\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_{k+1}+\ldots+b_{k'})\Gamma(b_{k'+1})\ldots\Gamma(b_n)}\left[1-(x_{r+1}+\ldots+x_n)\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\ldots+x_rE_r+x_{r+1}\Delta_{r+1}+\ldots+x_n\Delta_n}{1-(x_{r+1}+\ldots+x_n)}\right]^{-a}.$$

Further making an appeal to (2.2), (2.3) and (2.4), we obtain

(4.2) $C=\left[1-(x_{r+1}+\ldots+x_n)\right]^{(-a)}\,{}^{(k,k')}_{(1)}\phi^{(n)}_{AC}(a,a',b_{k'+1},\ldots,b_r,c_{r+1}-b_{r+1},\ldots,c_n-b_n;$

$$c_1,\ldots,c_n;\frac{x_1}{1-(x_{r+1}+\ldots+x_n)},\ldots,\frac{x_r}{1-(x_{r+1}+\ldots+x_n)},\frac{-x_{r+1}}{1-(x_{r+1}+\ldots+x_n)}$$

$$\left.\ldots\frac{-x_n}{1-(x_{r+1}+\ldots+x_n)}\right).$$

Now equating both values of C from (4.1) and (4.2), we derive $n-k'-1$ results in the following unified form

$$(4.3)\quad {}^{(k,k')}_{(1)}\phi^{(n)}_{AC}\left(a,a',b_{k'+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n\right)$$

$$=\left[-\left(x_{r+1}+\dots+x_n\right)\right]^{-a}\ {}^{(k,k')}_{(1)}\phi^{(n)}_{AC}\left(a,a',b_{k'+1},\dots,b_r,c_{r+1}-b_{r+1},\dots c_n-b_n;\right.$$

$$c_1,\dots,c_n;\frac{x_1}{1-\left(x_{r+1}+\dots+x_n\right)},\dots,\frac{x_r}{1-\left(x_{r+1}+\dots+x_n\right)},-\frac{x_{r+1}}{1-\left(x_{r+1}+\dots+x_n\right)}$$

$$\left.,\dots,\frac{-x_n}{1-\left(x_{r+1}+\dots+x_n\right)}\right),\ 1\le k\le k'\le r<n;\ k,k';r,n\in N,\ r=k',\dots,n-1.$$

Special Cases.

Particularly for $r=k'=k$, (4.3) reduces to Chandel ([8], p. (4.2)).

For $k'=k=0, r=0$ (4.3) reduces to ([8],p.(2.3)).

For $k'=k=0, r=1$ (4.3) reduces to ([8],p.(2.4)).

Similarly evaluating

$$\frac{\Gamma\left(c_{k+1}+\dots+c_{k'}\right)\Gamma\left(c_{k'+1}\right)\dots\Gamma\left(c_n\right)}{\Gamma\left(b_1\right)\dots\Gamma\left(b_n\right)}\left[1-\left(x_1E_1+\dots+x_nE_n\right)\right]^{-a}$$

$$\left\{\frac{\Gamma\left(b_1\right)\dots\Gamma\left(b_n\right)}{\Gamma\left(c_{k+1}+\dots+c_{k'}\right)\Gamma\left(c_{k'+1}\right)\dots\Gamma\left(c_n\right)}\right\}$$

in two ways, we finally derive $n-k'-1$ results in the following unified form

$$(4.4)\quad {}^{(k,k')}_{(2)}\phi^{(n)}_{AD}\left(a,b_1,\dots,b_n;c',c_{k'+1},\dots,c_n;x_1,\dots,x_n\right)$$

$$=\left(1-\left(x_{r+1}+\dots+x_n\right)\right)^{-a}\ {}^{(k,k')}_{(2)}\phi^{(n)}_{AD}\left(a,b_1,\dots,b_{k'+1},\dots,b_r,c_{r+1}-b_{r+1},\dots,\right.$$

$$c_n-b_n;c',c_{k'+1},\dots,c_r,c_{r+1},\dots,c_n;\frac{x_1}{1-\left(x_{r+1}+\dots+x_n\right)},\dots,\frac{x_{k'}}{1-\left(x_{r+1}+\dots+x_n\right)}$$

$$\left.,\dots,\frac{x_r}{1-\left(x_{r+1}+\dots+x_n\right)},\frac{-x_{r+1}}{1-\left(x_{r+1}+\dots+x_n\right)},\dots,\frac{-x_n}{\left[1-\left(x_{r+1}+\dots+x_n\right)\right]}\right),$$

$$r=k',\dots,n-1,\ 1\le k\le k'\le r<n;\ k,k',r,n\in N.$$

Special Case. For *k'=k*, (4.4) reduces to

$$(4.5)\quad {}^{(k)}_{(6)}\phi^{(n)}_{CD}\left(a,b_1,\dots,b_n;c_{k+1},\dots,c_n;x_1,\dots,x_n\right)$$

$$=\left[1-\left(x_{r+1}+\dots+x_n\right)\right]^{-a}\ {}^{(k)}_{(6)}\phi^{(n)}_{CD}\left(a,b_1,\dots,b_r,c_{r+1}-b_{r+1},\dots,c_n-b_n;\right.$$

$$c_{k+1},\dots,c_r,\dots,c_n;\frac{x_1}{1-\left(x_{r+1}+\dots+x_n\right)},\dots,\frac{x_k}{1-\left(x_{r+1}+\dots+x_n\right)},\dots,\frac{x_r}{1-\left(x_{r+1}+\dots+x_n\right)},$$

$$\frac{-x_{r+1}}{1-\left(x_{r+1}+\ldots+x_n\right)},\ldots,\frac{-x_n}{1-\left(x_1+\ldots+x_n\right)}\Bigg), r=k,\ldots,n-1,$$

$$1\le k\le r<n,\ \ k,r,n\in N,\ \ N=\{1,2,3,\ldots\},$$

where ${}^{(k)}_{(6)}\phi^{(n)}_{CD}$ is confluent multiple hypergeometric series introduced by Chandel and Vishwakarma [5].

Further considering and evaluating

$$\frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k'+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)}\left[1-(x_1E_1+\ldots+x_nE_n)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1)\ldots\Gamma(b_n)}{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k'+1})\ldots\Gamma(c_n)}\right\}$$

in two ways an samelines, we derive

$$(4.6)\quad {}^{(k,k')}_{(3)}\phi^{(n)}_{AD}\left(a,b_1,\ldots,b_n;c,c_{k'+1},\ldots,c_n;x_1,\ldots,x_n\right)$$

$$=\left[1-\left(x_{r+1}+\ldots+x_n\right)\right]^{-a}\,{}^{(k,k')}_{(3)}\phi^{(n)}_{AD}\Big(a,b_1,\ldots,b_{k'+1},\ldots,b_r,c_{r+1}-b_{r+1},\ldots,c_n-b_n;c,$$

$$c_{k'+1},\ldots,c_r,c_{r+1},\ldots,c_n;\frac{x_1}{1-\left(x_{r+1}+\ldots+x_n\right)},\ldots,\frac{x_{k'}}{1-\left(x_{r+1}+\ldots+x_n\right)}$$

$$,\ldots,\frac{x_r}{1-\left(x_{r+1}+\ldots+x_n\right)},\frac{-x_{r+1}}{1-\left(x_{r+1}+\ldots+x_n\right)},\ldots,\frac{-x_n}{1-\left(x_{r+1}+\ldots+x_n\right)}\Bigg),$$

$$r=k',\ldots,n-1,\ 1\le k\le k'\le r<n;\ \ k,k',r,n\in N.$$

Special Case. For $k'=k$, (4.6) gives the transformation of ${}^{(k)}F^{(n)}_{AD}$ due to Chandel ([8], p. (3.1)).

Here ${}^{(k,k')}F^{(n)}_{AC}$, ${}^{(k,k')}F^{(n)}_{AD}$ are two of generalized intermediate Lauricella functions and ${}^{(k,k')}_{(1)}\phi^{(n)}_{AC}$, ${}^{(k,k')}_{(2)}\phi^{(n)}_{AD}$ and ${}^{(k,k')}_{(3)}\phi^{(n)}_{AD}$ are their confluent forms introduced and studied by Chandel and Gupta ([6],[7]).

5. Transformations of Quadruple Hypergeometric functions. In this section, making an appeal to difference operators Δ and E, we derive transforamtion formulae for certain quadruple hypergeometric functions $K_2, K_8, K_{10}, K_{11}, K_{13}$ due to Exton ([10],[11]) and $F^{(4)}_{13}, F^{(4)}_{60}$ due to Sharma and Parihar [19].

Considering

$$\frac{\Gamma(d_1)\Gamma(d_2)\Gamma(d_3)\Gamma(d_4)}{\Gamma(b_1+b_2+b_3)\Gamma(b_4)}\left[1-(x_1E_1+x_2E_2+x_3E_3+x_4E_4)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+b_2+b_3)\Gamma(b_4)}{\Gamma(d_1)\Gamma(d_2)\Gamma(d_3)\Gamma(d_4)}\right\}$$

and employing same techniques of sections 3 and 4 in two ways, we derive

(5.1) $K_2(a,a,a,a,b,b,b,b_4;d_1,d_2,d_3,d_4;x_1,x_2,x_3,x_4)$

$$=(1-x_4)^{-a}K_2\left(a,a,a,a,b,b,b,b_4;d_1,d_2,d_3,d_4-b_4;\frac{x_1}{1-x_4},\frac{x_2}{1-x_4},\frac{x_3}{1-x_3},\frac{-x_r}{1-x_r}\right).$$

Similarly, Considering

$$\frac{\Gamma(d_1+d_2)\Gamma(e_3)\Gamma(e_4)}{\Gamma(b_1+b_r)\Gamma(c_3)\Gamma(c_4)}\left[1-(x_1E_1+x_2E_2+x_3E_3+x_4E_4)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+b_2)\Gamma(c_3)\Gamma(c_4)}{\Gamma(d_1+d_2)\Gamma(e_3)\Gamma(e_4)}\right\}$$

and applying the same techniques, we derive

(5.2) $K_8(a,a,a,a,c_1,c_1,c_3,c_4;e_1,e_1,e_3,e_4;x_1,x_2,x_3,x_4)$

$$=(1-x_3-x_4)^{-a}K_8\left(a,a,a,a,c_1,c_1,e_3-c_3,e_4-c_4;e_1,e_1,e_3,e_4;\right.$$

$$\left.\frac{x_1}{1-x_3-x_4},\frac{x_2}{1-x_3-x_4},\frac{-x_3}{1-x_3-x_4},\frac{-x_4}{1-x_3-x_4}\right).$$

Also considering

$$\frac{\Gamma(d_1)\Gamma(d_2)\Gamma(d_3)\Gamma(d_4)}{\Gamma(b_1+b_2)\Gamma(b_3)\Gamma(b_4)}\left[1-(x_1E_1+x_2E_2+x_3E_3+x_4E_4)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+b_2)\Gamma(b_3)\Gamma(b_4)}{\Gamma(d_1)\Gamma(d_2)\Gamma(d_3)\Gamma(d_4)}\right\}.$$

we establish

(5.3) $K_{10}(a,a,a,a,b,b,b_3,b_4;d_1,d_2,d_3,d_4;x_1,x_2,x_3,x_4)$

$$=(1-x_3-x_4)^{-a}K_{10}(a,a,a,a,b,b,d_3-b_3,d_4-b_4;d_1,d_2,d_3,d_4);$$

$$\left.\frac{x_1}{1-x_3-x_4},\frac{x_2}{1-x_3-x_4},\frac{-x_3}{1-x_3-x_4},\frac{-x_4}{1-x_3-x_4}\right).$$

Further Considering

$$\frac{\Gamma(e_1+e_2+e_3)\Gamma(d_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}\left[1-\left(x_1E_1+x_2E_2+x_3E_3+x_4E_4\right)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}{\Gamma(e_1+e_2+e_3)\Gamma(d_4)}\right\}$$

and applying the same technique in two ways, we obtain

(5.4) $K_{11}(a,a,a,a,b_1,b_2,b_3,b_4,c,c,c,d_4,x_1,x_2,x_3,x_4)$

$$=(1-x_4)^{-a}K_{11}\left(a,a,a,a,b_1,b_2,b_3,d_4-b_4;c,c,c,d_4;\right.$$

$$\left.\frac{x_1}{1-x_4},\frac{x_2}{1-x_4},\frac{x_3}{1-x_4},\frac{-x_4}{1-x_4}\right).$$

Now applying same technique on

$$\frac{\Gamma(c_1+c_2)\Gamma(d_3)\Gamma(d_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}\left[1-\left(x_1E_1+x_2E_2+x_3E_3+x_4E_4\right)\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}{\Gamma(c_1+c_2)\Gamma(d_3)\Gamma(d_4)}\right\},$$

we derive

(5.5) $K_{13}(a,a,a,a,b_1,b_2,b_3,b_4;c,c,d_3,d_4;x_1,x_2,x_3,x_4)$

$$=(1-x_3-x_4)^{-a}K_{13}\left(a,a,a,a,b_1,b_2,d_3-b_3,d_4-b_4;\right.$$

$$\left.\frac{x_1}{1-x_3-x_4},\frac{x_2}{1-x_3-x_4},\frac{x_3}{1-x_{3-}x_4},\frac{-x_4}{1-x_2-x_4}\right).$$

Similarly, considering

$$\frac{\Gamma(c_1+c_2)\Gamma(c_3)\Gamma(c_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\left[1-\left(x_1E_1+x_2E_2+x_3E_3+x_4E_4\right)\right]^{-a}$$

$$\left\{\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c_1+c_2)\Gamma(c_3)\Gamma(c_4)}\right\}$$

and applying the same techniques in two ways, we derive

(5.6) $F_{13}^{(4)}(a_1,a_2,a_3,a_4,a,a,a,a;c,c,c_3,c_4;x_1,x_2,x_3,x_4)$

$$=(1-x_3-x_4)^{-a}F_{13}^{(4)}\left(a_1,a_2,c_3-a_3,c_4-a_4,a,a,a,a,c,c,c_3,c_4;\right.$$

$$\left.\frac{x_1}{1-x_3-x_4};\frac{x_2}{1-x_3-x_4},\frac{-x_3}{1-x_3-x_4},\frac{-x_4}{1-x_3-x_4}\right).$$

Finally, Considering

$$\frac{\Gamma(c_1+c_2+c_3)\Gamma(c_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\left[1-(x_1E_1+x_2E_2+x_3E_3+x_4E_4)\right]^{-a}$$

$$\left\{\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c_1+c_2+c_3)\Gamma(c_4)}\right\}$$

and employing the same technique, we derive

(5.7) $F_{60}^{(4)}(a_1,a_2,a_3,a_4,a,a,a,a;c,c,c,c;x_1,x_2,x_3,x_4)$

$$=(1-x_4)^{-a}F_{60}^{(4)}\Big(a_1,a_2,a_3,c_4-a_4,a,a,a,a,c,c,c,c_4;$$

$$\frac{x_1}{1-x_3},\frac{x_2}{1-x_4},\frac{-x_3}{1-x_4},\frac{-x_4}{1-x_4}\Big).$$

ACKNOWLEDGEMENT

We wish to record our deepest and sincerest feelings of gratitude to Dr. R. C. Singh Chandel for his kind helf and guidance during the preparation of this paper.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPINGS OF TYPE (P) USING IMPLICIT RELATION IN INTUTITIONISTIC FUZZY METRIC SPACE

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(*Received : September* 17, 2015; *Final form : October* 20, 2015)

ABSTRACT

In this paper we prove a common fixed point theorem for compatible mapping of type (P) in Intuitionistic fuzzy metric space using impilict relation. Our result modifies the results of Koireng and Rohen [8].

2010 Mathematics Subject Classification: 54H25, 54E50

Keywords: Compatible maps, Fuzzy metric spaces, intuitionistic fuzzy metric space, Compatible maps of type (P), implicit relation.

1. Introduction. The concept of fuzzy sets was introduced initially by Zedeh [19], which laid the foundation of fuzzy mathematics. George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [9]. They also obtained that every metric space induces a fuzzy metric space. Sessa [14] proved a generalization of commutativity. Further Jungck [7] more generalized commutativity called Compatibility in metric space. In [3] Cho, Sharma et. al. introduced the concept of semi compatibility in D-metric space.

The first important result of compatible mapping was obtained by Jungck [7]. Pathak, Chang and Cho [11] introduced the concept of compatible mapping type (P). Coker [4] introduced the concept of intuitionistic topological spaces. Alaca et al. [1] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space as Park [12,13], with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [9] and proved the well known fixed point theorems of Banach [2] in the setting of intuitionistic fuzzy metric space. Later Turkoglu et al. [18] gave a generalization of Jungck's common fixed point theorem to intuitionistic fuzzy metric spaces. They first formulated the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [10].

Singh and Jain [15] proved various fixed point theorems using the concept of semi-compatibility, compatibiliy and imlicit relations in Fuzzy metric space. Recently Bijendra Singh et.al.[16,17] introduced the concept of compatible mapping in fuzzy metric spaces, fuzzy sets and systems and introduced the concept of semicompatible mapping in context of fuzzy metric sapce.

In the present paper, we prove a common fixed point theorem for compatible mapping of type (p) in Intuitionistic fuzzy metric space using implicit relation. Our result modifies the results of Koireng and Rohen [8].

2. Preliminaries and Definitions.

Definition 2.1 A mapping $T{:}X\rightarrow X$ is called *contractive* if $d(Tx,Ty)\le kd(x,y)$ for all $x, y\in X$.

Definition 2.2. A binary operation $\star{:}[0,1]\times[0,1]\rightarrow[0,1]$ is *continuous t-norm* if "$\star$" satisfies the following conditions :

(i) $\star$ is commutative and associative,

(ii) $\star$ is continuous,

(iii) $a\star 1=a$ for all $a\in[0,1]$,

(iv) $a\star b\le c\star d$ whenever $a\le c$ and $b\le d$, and $a,b,c,d\in[0,1]$.

Definition 2.3. A binary operation $\diamond{:}[0,1]\times[0,1]\rightarrow[0,1]$ is continuous t-conorm if "$\diamond$" satisfies the following conditions :

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a\diamond 0 = a$ for all $a \in [0,1]$;

(iv) $a\diamond b \leq c\diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.4. A 5-tuple $(X,M,N,\star,\diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M,N are fuzzy sets on $X^2\times(0, \infty)$ satisfying the following conditions: for all $x,y,z \in X, s,t > 0$:

(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$,

(IFM-2) $M(x,y,t) > 0$,

(IFM-3) $M(x,y,t) = 1$ if and only if $x=y$,

(IFM-4) $M(x,y,t) = M(y,x,t)$,

(IFM-5) $M(x,y,t)\star M(y, z, s) \leq M(x, z, t + s)$,

(IFM-6) $M(x, y, .) : (0,\infty)\to [0,1]$ is continuous,

(IFM-7) $N(x, y,t) > 0$,

(IFM-8) $N(x,y,t)=N(y,x,t)$,

(IFM-9) $N(x,y,t)\diamond N(y,z,s) \geq N(x,z,t+s)$,

(IFM-10) $N(x,y,.):(0,\infty)\to [0, 1]$ is continuous.

Then (M,N) is called an *intuitionistic fuzzy metric* on X.

Note. $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non nearness between x and y with respect to 't' respectively.

Remark 2.1. Every fuzzy metric space $(X,M,\star)$ is an intuitionistic fuzzy metric space of the form $(X,M,1-M,\star,\diamond)$ such that t-norm $\star$ and t-conorm $\diamond$ are associated [17],

i.e. $x\diamond y = 1-((1-x) \star(1-y))$ for any $x, y \in [0,1]$.

Remark 2.2. An intuitionistic fuzzy metric space $X,M(x,y,.)$ is non-decreasing and $N(x,y,.)$ is non-increasing for all $x,y \in X$.

Example 2.1. Let (X,d) be a metric space. Denote $a*b=ab$ and $a\diamond b=\min\{1,a+b\}$ for all $a,b \in [0,1]$. Let M and N be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$M(x,y,t) = \frac{t}{t+md(x,y)}, \quad N(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$$

in which $m>1$. Then $(X,M,N, \star, \diamond)$ is an intuitionistic fuzzy metric space.

Example 2.2. Let $X=N$. Define $a* b=\max\{0,a+b-1\}$ and $a.b=a+b-ab$ for all $a,b \in [0,1]$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$ as follows:

$$M(x,y,t)=\begin{cases} x/y & ; x\le y \\ y/x & ; y\le x \end{cases} \text{ and } N(x,y,t)=\begin{cases}(x-y)/y & ; x\le y \\ (x-y)/x & ; y\le x\end{cases}.$$

Remark 2.3. Note that, in the above example, the t-norm "$*$" and the t-conorm "$\diamond$" are not associated and there exists no metric d on X satisfying

$$M(x,y,t)=\frac{t}{t+md(x,y)}\quad,\quad N(x,y,t)=\frac{d(x,y)}{t+d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note that above function (M,N) is not an intuitionistic fuzzy metric with the t-*norm* and t-*conorm* defined as

$$a*b = \min\{a,b\} \text{ and } a\diamond b=\max\{a,b\}.$$

Definition 2.5. Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. Then

(i) a sequence $\{x_n\}$ in X is said to be *Cauchy sequence* if for each $\varepsilon>0$ and $t>0$, there exist $n_0\in N$ such that $M(x_n,x_m,t)> 1-\varepsilon$ and $N(x_n,x_m,t)<\varepsilon$ for all $n,m=n_0$.

(ii) a sequence $\{x_n\}$ in X is said to *converge* to x if for each $\varepsilon>0$ and $t>0$, there exists $n_0\in N$ such that $M(x_n,x,t)>1-\epsilon$ and $N(x_n,x,t)<\varepsilon$ for all $n=n_0$.

(iii) $(X,M,N,*,\diamond)$ is called *complete intuitionistic fuzzy metric space* if every Cauchy sequence is convergent in it.

Throughout this paper, $(X,M,N,*,\diamond)$ will denote the *intuitionistic fuzzy metric space* in the sense of *Definition* (2.3) with the following condition:

(IFM-11) $\lim_{n\to\infty} M(x,y,t)=1$ for all $x,y\in X$, $t> 0$.

Now, we give the concept of *commutativity in intuitionistic fuzzy metric spaces.*

Definition 2.6. Let A and B be maps from an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ into itself. The maps A and B are said to be commutative, if $ABx_n=BAx_n$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n=Lim_{n\to\infty} Bx_n=z$ for some $z\in X$.

The following definition was introduced by Turkogulu et al. [18].

Definition 2.7. Let A and B be maps from a intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ into itself. The maps A and B are said to be compatible if, for all $t > 0$,

$\lim t_{n\to\infty} M(ABx_n,BAx_n,t)=1$ whenever $\{x_n\}$ is a sequence in X such that

$\lim t_{n\to\infty} Ax_n = \lim t_{n\to\infty} Bx_n = z$ for some $z \in X$.

Proposition 2.1. In intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ limit of a sequence is unique.

Proof. Let if possible $\{x_n\} \to x$ and $\{x_n\} \to y$ then $\lim_{n\to\infty} M(x_n,x,t) = 1 = \lim_{n\to\infty} M(x_n,y,t)$ and $\lim_{n\to\infty} N(x_n,x,t) = 0 = \lim_{n\to\infty} N(x_n,y,t)$.

Now $\lim_{n\to\infty} M(x_n,x,t) = 1 = \lim_{n\to\infty} M(x_n,y,t)$ and $\lim_{n\to\infty} N(x_n,x,t) = 0 = \lim_{n\to\infty} N(x_n,y,t)$

Then $M(x_n,x,t) \geq M(x,x_n,t/2) * M(y,x_n,t/2)$ and $N(x_n,x,t) \leq N(x,x_n,t/2) * N(y,x_n,t/2)$.

Taking $\lim_{n\to\infty}$, $M(x,y,t) \geq 1*1$ and $z \in X$

i.e. $M(x,y,t) = 1$ and $N(x,y,t) = 0$ for all $t>0$, thus $x=y$.

Hence limit is unique.

Proposition 2.2 (A,S) is a semi-compatible pair of self maps of a intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ and S is continuous then (A,S) is compatible.

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to x$ and $\{Sx_n\} \to x$, by semi compatibility of (A,S) we have $\lim_{n\to\infty} ASx_n \to Sx$. As S is continuous we get $\lim_{n\to\infty}(SAx_n, ASx_n, t) = M(Sx,Sx,t) = 1$and $\lim_{n\to\infty}(SAx_n, ASx_n, t) = N(Sx,Sx,t) = 0$.

Hence (A,S) is compatible.

Note. Converse of this proposition is not true.

Definition 2.8. Self mappings A and S of a intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is said to be *compatible of type* (P) if $\{x_n\} \to x$ then $\lim_{n\to\infty} M(AAx_n, SSx, t) = 1$ and $\lim_{n\to\infty} N(AAx_n, SSx, t) = 0$ for all $t>0$, Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for all $z \in X$.

Lemma [1]. Let $(X,M,N,*,\diamond)$ be a intuitionistic fuzzy metric space. If there exists $k \in X$ such that $M(x, y, kt) \geq M(x, y, t/k_n)$ and $N(x, y, kt) \leq N(x, y, t/k_n)$ for positive integer n, taking $\lim_{n\to\infty}$, $M(x,y,kt) \geq 1$. Hence $x=y$.

Proposition 2.3. Let $(X,M,N,*,\diamond)$ be a intuitionistic fuzzy metric space and let A and S be continuous mappings of X then A and S are compatible if and only if they are compatible of type (P).

Proposition 2.4. Let $(X,M,N,*,\diamond)$ be a intuitionistic fuzzy metric space and let A and S be compatible mappings of type (P) and $Az=Sz$ for some $z \in X$, then $AAz=ASz=SAz=SSz$.

Proposition 2.5. Let $(X,M,N,*,\diamond)$ be a intuitionistic fuzzy metric space and let A and S be compatible mappings of type (P) and let $A,S \to z$ as $n \to \infty$ for some $z \in X$. then

(i) $\lim_{n\to\infty} SSx_n = Az$ if A is continuous at z

(ii) $\lim_{n\to\infty} AAx_n = Sz$, if S is continuous at z

(iii) $ASz = SAz$ and $Az = Sz$ if A and S are continuous at z.

Definition 2.9. A Class of Implicit Rrelation.

Let φ be the set of all real and continuous functions from $\varphi:[0,1]^s \to R$ satisfying the following conditions:

(i) $(A\text{-}1)\varphi$ is non decreasing if non-increasing in second, third, fourth and fifth argument.

(ii) $(A-2)$ $\varphi(u,v,v,u,v) \geq 0 \Rightarrow u \geq v$

(iii) $\varphi(u,v,v,v,v) \geq 0 \Rightarrow u \geq v$.

Example. $\phi(t_1,t_2,t_3,t_4,t_5) = t_1 - Max(t_2,t_3,t_4,t_5)$.

3. Main Result.

Theorem 3.1. Let A,B,S and T be self mappings of a complete intuitionistic fuzzy metric space with continuous t-norm defined by $a * b = \min\{a,b\}$, where $a,b \in [0,1]$ satisfying

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(ii) S and T are continuous,

(iii) Pairs (A,S) and (B,T) are compatible of type (P) ,

(iv) $\exists$ some $k \in (0,1)$ such that for all $x,y \in X, t > 0$

$$\phi(M(Ax,By,kt), M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,kt), M(Ty,Ax,t)) \geq 0$$

and

$$\phi(N(Ax,By,kt), N(Sx,Ty,t), N(Sx,Ax,t), N(Ty,By,kt), N(Ty,Ax,t)) \leq 0$$

(v) $\forall x,y \in X, M(x,y,t) \to 1$ as $t \to \infty$.

Then A,B,S and T have a unique common fixed point.

Proof. $x_0 \in X$ be any point as $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\exists x_1 \in X$ and $x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively we construct a sequence $\{y_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n-1}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$; $(y_{2n} = Sx_{2n}), n = 0,1$ with $x = x_{2n}$, $y = x_{2n+1}$.

Using contractive condition, we get

$$\phi(M(Ax_{2n}, Bx_{2n+1}, kt), M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, kt), M(Tx_{2n+1}, Ax_{2n}, t)) \geq 0$$

and

$$\phi(N(Ax_{2n}, Bx_{2n+1}, kt), N(Sx_{2n}, Tx_{2n+1}, t), N(Sx_{2n}, Ax_{2n}, t), N(Tx_{2n+1}, Bx_{2n+1}, kt), N(Tx_{2n+1}, Ax_{2n}, t)) \leq 0$$

$$\Rightarrow \phi(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+1}, t)) \geq 0$$

and

$$\phi(N(y_{2n+1}, y_{2n+2}, kt), N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n+2}, t), N(y_{2n+1}, y_{2n+2}, kt), N(y_{2n+1}, y_{2n+1}, t)) \leq 0$$

$$\Rightarrow \lim_{n\to\infty} M(y_{n+p}, y_n, t) = 1 \text{ and } \lim_{n\to\infty} N(y_{n+p}, y_n, t) = 0 \;;\forall\, p \text{ and } t > p.$$

Since ϕ is non decreasing in fifth argument therefore

$$\phi(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+1}, t) \geq 0$$

and

$$\phi(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+1}, t)) \geq 0.$$

Therefore by property (ii) of implicit relation

$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t)$ and $N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n+1}, y_{2n}, t)$.

Similarly

$M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}, y_{2n-1}, t)$ and $N(y_{2n+1}, y_{2n}, kt) \leq N(y_{2n}, y_{2n-1}, t)$.

Hence

$M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$ and $N(y_{n+1}, y_n, kt) \leq N(y_n, y_{n-1}, t)$ $\forall n$.

We show that

$\lim_{n\to\infty} M(y_{n+p}, y_n, t) = 1$ and $\lim_{n\to\infty} N(y_{n+p}, y_n, t) = 0$;$\forall\, p$ and $t > p$.

Now

$$M(y_{n+1}, y_n, t) \geq M(y_n, y_{n-1}, t/k) = M(y_n, y_{n-2}, t/k^2) = \ldots$$

and

$$N(y_{n+1}, y_n, t) < N(y_n, y_{n-1}, t/k) = N(y_n, y_{n-2}, t/k^2) = \ldots$$

Thus the result holds for p=1. By induction hypothesis suppose that the result holds for $p = r$, therefore

$$M(y_n, y_{n+r+1}, t) \geq M(y_n, y_{n+r}, t/2) * M(y_{n+r}, y_{n+r+1}, t/2) \to 1*1 = 1$$

and

$$N(y_n, y_{n+r+1}, t) < N(y_n, y_{n+r}, t/2) * N(y_{n+r}, y_{n+r+1}, t/2) \to 0*0 = 0.$$

Thus result holds for $p=r+1$.

Hence $\{y_n\}$ is a Cauchy sequence in X, and as X is complete hence we get $\{y_n\} \to z \in X$. Hence

$Ax_{2n} \to z$, $Sx_{2n} \to z$ ...(i)

$Tx_{2n+1} \to z$, $Bx_{2n+1} \to z$. ...(ii)

Since pairs (A,S) and (B,T) are compatible of type (P) therefore from proposition we get

$$AAx_{2n} \to Sz , SSx_{2n} \to Az, BBx_{2n} \to Tz, TTx_{2n+1} \to Bz.$$

From contractive condition, we get

$\phi(M(Sz,Tz,kt), M(Sz,Tz,t), M(Sz,Sz,t)M(Tz,Tz,kt), M(Sz,Tz,t)) \geq 0$

and

$\phi(N(Sz,Tz,kt), N(Sz,Tz,t), N(Sz,Sz,t)N(Tz,Tz,kt), N(Sz,Tz,t)) \leq 0$

$\Rightarrow \phi(M(Sz,Tz,kt), M(Sz,Tz,t), 1, 1, (Sz,Tz,t)) \geq 0$ and

$\phi(N(Sz,Tz,kt), N(Sz,Tz,t), 1, 1, (Sz,Tz,t)) \leq 0$

$\Rightarrow M(Sz,Tz,kt) \geq M(Sz,Tz,t)$ and $N(Sz,Tz,kt) \leq N(Sz,Tz,t)$

$\Rightarrow M(Sz,Tz,kt) \geq M(Sz,Tz,t)$ and $N(Sz,Tz,kt) \leq N(Sz,Tz,t)$

$\Rightarrow Sz = Tz$ (by Lemma).

From contractive condition

$\phi(M(Az, BTx_{2n+1}, kt), M(Sz, TTx_{2n+1}, t), M(Az,Sz,t)M(BTx_{2n+1}, TTx_{2n+1}, t),$
$M(TTx_{2n+1}, Az, t)) \geq 0$

and

$\phi(N(Az, BTx_{2n+1}, kt), N(Sz, TTx_{2n+1}, t), N(Az,Sz,t), N(BTx_{2n+1}, TTx_{2n+1}, t),$
$N(TTx_{2n+1}, Az, t)) \leq 0$. As $n \to \infty$

$\phi(M(Az,Tz,kt), M(Sz,Sz,t), M(Az,Tz,t), M(Tz,Tz,kt), M(Az,Tz,t)) \geq 0$ and
$\phi(N(Az,Tz,kt), N(Sz,Sz,t), N(Az,Tz,t), N(Tz,Tz,kt), N(Az,Tz,t)) \leq 0$

$\Rightarrow \phi(M(Az,Bz,kt), M(Az,Tz,t), M(Az,Tz,t), M(Az,Tz,kt), M(Az,Tz,t)) \geq 0$

and

$\phi(N(Az,Bz,kt), N(Az,Tz,t), N(Az,Tz,t), N(Az,Tz,kt), N(Az,Tz,t)) \leq 0$

(Since φ is non-increasing in second and fourth argument)

$\Rightarrow M(Az,Tz,kt) \geq M(Az,Tz,t)$ and $N(Az,Tz,kt) \leq N(Az,Tz,t)$

$\Rightarrow Az = Tz = Sz.$ [By Lemma-1].

Again from contractive condition

$\phi(M(Az,Bz,kt), M(Sz,Tz,t), M(Az,Sz,t), M(Tz,Bz,kt), M(Tz,Az,t)) \geq 0$

and $\phi(N(Az,Bz,kt), N(Sz,Tz,t), N(Az,Sz,t), N(Tz,Bz,kt), N(Tz,Az,t)) \leq 0$

$\Rightarrow \phi(M(Az,Bz,kt),M(Az,Az,t),M(Az,Az,t),M(Az,Bz,kt),M(Az,Az,t)) \geq 0$
and $\phi(N(Az,Bz,kt),N(Az,Az,t),N(Az,Az,t),N(Az,Bz,kt),N(Az,Az,t)) \geq 0$
$\Rightarrow \phi(M(z,Bz,kt),M(z,Tz,t),M(z,z,t),M(Tz,Bz,kt),M(z,Tz,t)) \geq 0$
and $\phi(N(z,Bz,kt),N(z,Tz,t),N(z,z,t),N(Tz,Bz,kt),N(z,Tz,t)) \leq 0$.
Since φ is non increasing in second, third and fifth argument
$\Rightarrow \phi\left(M\left(Az,Bz,kt\right),n\left(Az,Bz,t\right),M\left(Az,Bz,t\right),M\left(Az,Bz,kt\right),M\left(Az,Bz,t\right)\right) \geq 0$
$\phi(M(Az,Bz,kt),N(Az,Bz,t),M\left(Az,Bz,t\right),M(Az,Bz,kt),M(Az,Bz,t)) \geq 0$
and $\phi(N(Az,Bz,kt),N(Az,Bz,t),N(Az,Bz,t),N(Az,Bz,kt),N(Az,Bz,t)) \leq 0$
$\Rightarrow M(Az,Bz,kt) \geq M(Az,Bz,t)$ and $N(Az,Bz,kt) \leq N(Az,Bz,t)$
$\Rightarrow Az=Bz$
$\Rightarrow Az=Tz=Sz=Bz$ and now we show that $Bz=z$.
By contractive condition
$\phi(M(z,Bz,kt),M(z,Tz,t),M(z,z,t),M(Tz,Bz,kt),M(z,Tz,t)) \geq 0$
and $\phi(N(z,Bz,kt),N(z,Tz,t),N(z,z,t),N(Tz,Bz,kt),N(z,Tz,t)) \leq 0$
$\Rightarrow \phi(M(z,Bz,kt),M(z,Bz,t),1,1,M(Bz,z,t)) \geq 0$
and $\phi(N(z,Bz,kt),N(z,Bz,t),1,1,N(Bz,z,t)) \leq 0$.
Since φ is non increasing in third and fifth argument
$\phi(M(z,Bz,kt),M(z,Bz,t),M(z,Bz,t),M(z,Bz,t),M(z,Bz,t)) \geq 0$
and $\phi(N(z,Bz,kt),N(z,Bz,t),N(z,Bz,t),N(z,Bz,t),N(z,Bz,t)) \leq 0$
$\Rightarrow M(z,Bz,kt) \geq M(Bz,z,t)$ and $N(z,Bz,kt) \leq N(Bz,z,t)$ [By Lemma-1]
$\Rightarrow Bz=z$.
Hence
$Az=Bz=Sz=Tz=z$.
Thus z is a common fixed point of A,B,S, and T.

Uniqueness. Let z and z' be two common fixed points of the maps A,B,S and T. Then $Az=Bz=Tz=Sz=z$ and $Az'=Bz'=Tz'=Sz'=z'$
Using contractive condition, we get
$\phi(M(Az,Bz',kt),M(Sz,Tz',t),M(Sz,Az,t),M(Tz',Bz',kt),M(Tz',Az,t)) \geq 0$
and $\phi(N(Az,Bz',kt),N(Sz,Tz',t),N(Sz,Az,t),N(Tz',Bz',kt),N(Tz',Az,t)) \leq 0$
$\Rightarrow \phi(M(z,z',kt),M(z,z',t),M(z,z,t),M(z',z',kt),M(z',z,t)) \geq 0$
and $\phi(N(z,z',kt),N(z,z',t),N(z,z,t),N(z',z',kt),N(z',z,t)) \leq 0$
$\Rightarrow \phi(M(z,z',kt),M(z,z',t),M(z,z',t),M(z',z,t),M(z',z,t)) \geq 0$
and $\phi(N(z,z',kt),N(z,z',t),N(z,z',t),N(z',z,t),N(z',z,t)) \leq 0$.

$\Rightarrow \phi(M(z,z',kt),M(z,z',t),M(z,z',t),M(z',z',t)) \geq 0$

and $\phi(N(z,z',kt),N(z,z',t),N(z,z',t),N(z',z',t)) \geq 0$.

(since ϕ is non increasing in third and fourth arangement)

$\Rightarrow M(z,z',kt) \geq M(z,z',t)$ and $N(z,z',kt) \leq N(z,z',t)$ [by second condition]

$\Rightarrow z = z'$ [by Lema1].

Hence z is a unique common fixed point of maps A,B,S,T.

Corollary. Let A,B,S and T be self mappings of a complete fuzzy metric space with continuous t-norm defined by above satisfying (i), (ii), (iii), (v) of Theorem 3.1 and there exist some $k \in (0,1)$ such that for $x,y \in X, t > 0$.

$$\phi\big(M(Az,By,kt),M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,2t),M(Ty,Ax,t)\big) \geq 0$$

and $\phi\big(N(Az,By,kt),N(Sx,Ty,t),N(Sx,Ax,t),N(Ty,By,2t),N(Ty,Ax,t)\big) \leq 0.$

Then A,B,S and T have a unique common fixed point.

Theorem 3.2. Let $(X,M,N,*,\diamond)$ be a complete fuzzy metric spaces. S and T have a common fixed point in X if and only if there exists a self mapping A of X such that the following conditions are satisfied:

(i) $A(X) \subset T(X) \cap S(X)$

(ii) pairs (A,S) and (A,T) are compatible of type (P)

(iii) $\exists k \in (0,1)$ such that for all $x,y \in X$, $t > 0$,

$$\phi\big(M(Ax,Ay,kt),M(Sx,Ty,t),M(Ax,Sx,t),M(Ty,Ay,kt),M(Ty,Ax,t)\big) \geq 0$$

and

$$\phi\big(N(Ax,Ay,kt),N(Sx,Ty,t),N(Ax,Sx,t),N(Ty,Ay,kt),N(Ty,Ax,t)\big) \leq 0.$$

Then A,B,S and T have a unique common fixed point.

Proof. We have shown that the necessity of the conditions (i)-(iii). Suppose that S and T Have a common fixed point in X, say z. Then $z=Tz$. Let $Ax=z$ *for* all $x \in X$. Then we have $A(X) \subset T(X) \cap S(X)$ and we know that $[A,S]$ and $[A,T]$ are compatible mappings of type (P), in fact $AS = SA$ and $AT=TA$, and hence the conditions (i) and (ii) are satisfied. For some $p \in (0,1)$, we get $M(Ax,Ay,kt)=1$and $N(Ax,Ay,kt)=0$. So

(iv) $\phi\big(M(Ax,Ay,kt),M(Sx,Ty,t),M(Ax,Sx,t),M(Ty,Ay,kt),M(Ty,Ax,t)\big) \geq 0$

and $\phi\big(N(Ax,Ay,kt),N(Sx,Ty,t)N(Ax,Sx,t),N(Ty,Ay,kt),N(Ty,Ax,t)\big) \leq 0$

for all $x,y \in X$, $t > 0$, and hence condition (iii) is satisfied.

Now for the sufficiency of conditions, let $A=B$ in Theorem 3.1. Then A,S and T have a common fixed point in X.

4. Conclusion. In this paper we used the concept compatible self mappings of type (p) in intuitionistic fuzzy metric space and proved existence and uniqueness of fixed point under the conditions of class of implicit relation.

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(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

COMMON FIXED POINT THEOREM FOR *T*-CONTRACTION IN CONE METRIC SPACES UNDER RATIONAL CONDITION

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(Received : January 10, 2015; *Revised final form : August* 12, 2015)

ABSTRACT

In this paper we prove a common fixed point theorem for *T*-contraction of two maps in cone metric space for rational expression. The result of this paper extends and generalizes well-known comparable results in the literature.

2010 Mathematics Subject Classification : 54H25, 47H10

Keywords : Common fixed point, Contractgive type mapping, Cone metric space, *T*-Contraction principle.

1. Introduction. The Banach contraction principle is a very popular and effective tool in solving existence problems in many branches of mathematical analysis. The Banach contraction principle with rational expressions have been expanded and some fixed and common fixed point theorems have been obtained in [1],[2]. Haung and Zhang [6] initiated cone metric spaces, which is generalization of metric spaces, by substituting the real numbers with ordered Banach spaces. They have considered convergence in cone metric spaces, introduced completeness of cone metric spaces and proved a Banach contraction mapping theorem and some other

fixed point theorems involving contractive type mappings in cone metric spaces using the normality conditions. Several fixed and common fixed point results on cone metric spaces are derived in [7,4,11] and also various authors porved some common fixed point theorems with normal and non-normal cones in these space [5,10]. In this paper we prove the common fixed point theorem for T-contracation of two maps in cone metric space for rational expression in normal cone settings. Our result extends the main result of Dass and Gupta [3], Imdad-Khan [8] and Olaleru[9].

2. Preliminaries.

Definition 2.1. Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

(iii) If $x \in P$ and $-x \in P$ then $x = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$.

We shall write $x < y$ to mean $x \leq y$ and $x \neq y$. Also, we write $x << y$ if and only if $y - x \in intP$.

Definition 2.2. Let X be a non-empty set. Suppose the mapping $d: X \times X \rightarrow E$ stisfies

(i) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 2.3. Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(ii) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} d(x_n, x) = \theta$

(iii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$. we denote this by $\lim_{n,m \to \infty} d(x_n, x_m) = \theta$.

Definition 2.4. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent.

Lemma 2.5. Let (X,d) be a cone metric space over an ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric spaces

(P_1) If $x \leq y$ and $y \leq z$ then $x \leq z$.

(P_2) If $\theta \leq x << c$ for each $c \in intP$, *then* $x = \theta$

(P_3) If $x \in \lambda x$ where $x \in P$ and $0 \leq \lambda < 1$ then $x = \theta$

(P_4) Let $x_n \to \theta$ in E and $\theta << c$. Then there exists a positive integer n_0 such that $x_n << c$ for each $n > n_0$.

3. Main Result. Suppose that (X,d) is complete cone metric space, P is a solid cone and $T:X \to X$ is a continuous and one-to-one mapping. Moreover, let f and g be two mappings of X satisfying

$$d(Tfx,Tgy) \leq \alpha d(Tx,Ty) + \frac{\beta d(Tx,Tfx)d(Ty,Tgy)}{d(Tx,Tgy)+d(Ty,Tfx)+d(Tx,Ty)}.$$

For all $x, y \in X$ such that $x \neq y$, $d(Tx,Tgy)+d(Ty,Tfx)+d(Tx,Ty) \neq 0$, where α, β are non-negative real with $\alpha + \beta < 1$ and f and g are T-Contractions. Then

(i) There exists $z \in X$ such that $\lim_{n \to \infty} Tfx_{2n} = \lim_{n \to \infty} Tgx_{2n+1} = z$,

(ii) If T is sub sequentially convergent then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.

(iii) There exists a unique $w \in X$ such that $fw = gw = w$ that is f and g have a unique common fixed point.

(iv) It T is sequentially convergent then the sequence $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w.

Proof. Suppose that x_0 is an arbitrary point of X and define $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$, $n = 0,1,2...$ then

$$d(Tx_{2n+1},Tx_{2n+2}) = d(Tfx_{2n},Tgx_{2n+1})$$

$$\leq \alpha d(Tx_{2n},Tx_{2n+1}) + \frac{\beta d(Tx_{2n},Tfx_{2n})d(Tx_{2n+1},Tgx_{2n+1})}{d(Tx_{2n},Tgx_{2n+1})+d(Tx_{2n+1},Tfx_{2n})+d(Tx_{2n},Tx_{2n+1})}$$

$$\leq \alpha d(Tx_{2n},Tx_{2n+1}) + \frac{\beta d(Tx_{2n},Tx_{2n+1})d(Tx_{2n+1},Tx_{2n+2})}{d(Tx_{2n},Tx_{2n+2})+d(Tx_{2n+1},Tx_{2n+1})+d(Tx_{2n},Tx_{2n+1})}$$

$$\leq \alpha d(Tx_{2n},Tx_{2n+1}) + \frac{\beta d(Tx_{2n},Tx_{2n+1})d(Tx_{2n+1},Tx_{2n+2})}{d(Tx_{2n},Tx_{2n+2})+d(Tx_{2n},Tx_{2n+1})}.$$

By triangle inequality

$$d(Tx_{2n+1}, Tx_{2n+2}) \le d(Tx_{2n+1}, Tx_{2n}) + d(Tx_{2n}, Tx_{2n+2})$$

Thus

$$d(Tx_{2n+1}, Tx_{2n+2}) \le \alpha d(Tx_{2n}, Tx_{2n+1}) + \beta d(Tx_{2n}, Tx_{2n+1})$$
$$\le (\alpha+\beta) d(Tx_{2n}, Tx_{2n+1}).$$

Similarly

$$d(Tx_{2n+2}, Tx_{2n+3}) = d(Tfx_{2n+1}, Tgx_{2n+2})$$
$$\le \alpha d(Tx_{2n+1}, Tx_{2n+2}) + \frac{\beta d(Tx_{2n+1}, Tfx_{2n+1}) d(Tx_{2n+2}, Tgx_{2n+2})}{d(Tx_{2n+1}, Tgx_{2n+2}) + d(Tx_{2n+2}, Tfx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})}$$
$$\le \alpha d(Tx_{2n+1}, Tx_{2n+2}) + \frac{\beta d(Tx_{2n+1}, Tx_{2n+2}) d(Tx_{2n+2}, Tx_{2n+3})}{d(Tx_{2n+1}, Tx_{2n+3}) + d(Tx_{2n+2}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+2})}$$
$$\le \alpha d(Tx_{2n+1}, Tx_{2n+2}) + \frac{\beta d(Tx_{2n+1}, Tx_{2n+2}) d(Tx_{2n+2}, Tx_{2n+3})}{d(Tx_{2n+1}, Tx_{2n+3}) + d(Tx_{2n+1}, Tx_{2n+2})}.$$

By triangle inequality

$$d(Tx_{2n+2}, Tx_{2n+3}) \le d(Tx_{2n+2}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+3}).$$

Therefore

$$d(Tx_{2n+2}, Tx_{2n+3}) \le \alpha d(Tx_{2n+1}, Tx_{2n+2}) + \beta d(Tx_{2n+1}, Tx_{2n+2})$$
$$\le (\alpha+\beta) d(Tx_{2n+1}, Tx_{2n+2}).$$

If $\lambda = \alpha + \beta < 1$ then

$$d(Tx_n, Tx_{n+1}) \le \lambda d(Tx_{n-1}, Tx_n) \le \lambda^2 d(Tx_{n-2}, Tx_{n-1}) \le \ldots \le \lambda^n d(Tx_0, Tx_1).$$

Now for any $m > n$ and $\lambda < 1$

$$d(Tx_n, Tx_m) \le d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \ldots + d(Tx_{m-1}, Tx_m)$$
$$\le (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) d(Tx_0, Tx_1)$$
$$\le \frac{\lambda^n}{1-\lambda} d(Tx_0, Tx_1) \to \theta \text{ as } n \to \infty.$$

Thus from P_4 we have $\frac{\lambda^n}{1-\lambda} d(Tx_0, Tx_1) \ll c$ for all n sufficiently large and $\theta \ll c$. From (P_1) we have $d(Tx_n, Tx_m) \ll c$. It follows that $\{Tx_n\}$ is a

Cauchy sequence. Since a cone metric space X is complete, there exists $z \in X$ such that $Tx_n \to z$ as $n \to \infty$.

Thus $\lim_{n\to\infty} Tfx_{2n} = z$ and $\lim_{n\to\infty} Tgx_{2n+1} = z$.

Now if T is sub sequentially convergent $\{fx_{2n}\}(resp.\{gx_{2n+1}\})$ has a convergent subsequence. Thus there exists $w_1 \in X$ and $\{fx_{2ni}\}$ (resp. $w_2 \in X$ and $\{gx_{2n_{i+1}}\}$ such that

$$\lim_{n\to\infty} fx_{2n_i} = w_1 \quad \text{and} \quad \lim_{n\to\infty} gx_{2n_{i+1}} = w_2 .$$

Because of the continuity of T, we have

$$\lim_{n\to\infty} Tfx_{2n_i} = Tw_1 \quad \text{and} \quad \lim_{n\to\infty} Tgx_{2n_{i+1}} = Tw_2 .$$

Since T is injective there exists $w \in X$ such that $Tw=Tz$.

Now we shall show that $fw=gw=w$.

Let on contrary that $Tw \neq Tgw$ so that $d(Tw,Tgw) = w > 0$.

Now we can write

$$\begin{aligned}
w &\leq d(Tw,Tx_{2n+2}) + d(Tx_{2n+2},Tgw) \\
&\leq d(Tw,Tx_{2n+2}) + d(Tx_{2n+1},Tgw) \\
&\leq d(Tw,Tx_{2n+2}) + \alpha d(Tx_{2n+1},Tw) \\
&\quad + \frac{\beta d(Tx_{2n+1},Tfx_{2n+1})\, d(Tw,Tgw)}{d(Tx_{2n+1},Tgw) + d(Tw,Tfx_{2n+1}) + d(Tx_{2n+1},Tw)} \\
&\leq d(Tw,Tx_{2n+2}) + \alpha d(Tx_{2n+1},Tw) \\
&\quad + \frac{\beta d(Tx_{2n+1},Tfx_{2n+2})\, d(Tw,Tgw)}{d(Tx_{2n+1},Tgw) + d(Tw,Tfx_{2n+2}) + d(Tx_{2n+1},Tw)},
\end{aligned}$$

which on making $n \to \infty$ gives $d(Tw,Tgw) = 0$, a contradiction so that $Tw = Tgw$.

Since T is one-to-one therefore $gw=w$.

Similarly we can prove that $fw=w$. Thus $fw=gw=w$ that is w is a common fixed point of f and g.

Uniqueness. Now we shall show that w is the unique common fixed point.

Suppose that w^* is another common fixed point of f and g. Then

$d(Tw,Tw^*) = d(Tfw,Tgw^*)$

$$\leq \alpha d(Tw,Tw^*) + \frac{\beta d(Tw,Tfw)d(Tw^*,Tgw^*)}{d(Tw,Tgw^*)+d(Tw^*,Tfw)+d(Tw,Tw^*)}.$$

Using (P_3) it follows that $d(Tw,Tw^*)=\theta$ which implies the equality $Tw=Tw^*$. Since T is one-to-one $w=w^*$. Thus f and g have a unique common fixed point.

Ultimately if T is sequentially convergent, then we can replace n by n_i. Thus we have

$$\lim_{n\to\infty} fx_{2n} = w \qquad \text{and} \qquad \lim_{n\to\infty} gx_{2n+1} = w$$

Therefore if T is sequentially convergent, then the sequence $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converges to w.

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Jñānābha, Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

SUBCLASSES OF SPIRALLIKE FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

By

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(Received : September 05, 2015)

ABSTRACT

In this paper we introduce and study a new subclass $S^{\lambda}(\alpha,\beta,\gamma)$ of spirallike function defined by Ruscheweyh derivative [10]. We give a representation formula for the class $S^{\lambda}(\alpha,\beta,\gamma)$ some coefficient inequalities are discussed and η-spiral radius is obtained for $S^{\lambda}(\alpha,\beta,\gamma)$. Finally we prove a subordination theorem.

2010 Mathematics Subject Classification : Primary 30C45, Secondary 30C50

Keywords : Spirallike function, Ruscheweyh derivatives, η-spiral radius, Subordination theorem.

1. Introduction. A *logarithmic spiral* is a curve in the complex plane of the form

$$w = w_0 c^{-\lambda t}. -\infty < t < \infty \tag{1.1}$$

where w_0 and λ are complex constants with $w_0 \neq 0$ and $\text{Re}(\lambda) \neq 0$. If we take $\lambda = e^{i\lambda}$, $-\pi/2 < \lambda < \pi/2$, the curve (1.1) is called a λ-spiral which joins a given point $w_0 \neq 0$ to the origin. Observe that 0 spirals are radial half-lines.

A domain D containing the origin is said to be λ-*spirallike* if for each point $w_0 \neq 0$ in D the are of the λ-spiral from w_0 to the origin lies entirely in D. This obviously implies that D is simply connected. Let U denotes the open unit disk. A function f analytic in U, with $f(0)=0$ is said to be λ-spiral if its range is λ-spirallike. Finally such a function is spirallike if it is λ-spirallike for some λ.

Spirallike functions can be characterized by an analytic condition which is a slight generalization of the condition for starlikeness as follows: A function f analytic in U, with $f(0)=0$, $f'(0)\neq 0$ and $f(z)\neq 0$ for $0<|z|<1$ is said to be λ -spirallike [15] if

$$\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}>0, z\in U, \tag{1.2}$$

for some real $\lambda, |\lambda|<\pi/2$, Let S^{λ} denotes the class of all such functions. The class of spirallike functions was introduced and studied by Spaĉék [15] in 1933. Spaĉé also determined that spirallike functions which are univalent in U.

The notion of λ-spirallike functions generalizes the concept of starllike functions using logarithmic spirals instead of line segments. For instance, 0-spirallike functions are simply the starlike functions.

For the proof and geometric interpretation of condition (1.2), the reader may consult Duren ([4],2.7). We remark that many authors adopt the condition with $e^{-i\lambda}$ instead of $e^{i\lambda}$ in (1.2) as the definition of λ-spirallike functions.

In 1967, Libera [6] introduced the class of λ-spirallike functions of order $\alpha(0\leq\alpha\leq 1)$ in the open unit disk U, denoted by $S^{\lambda}(\alpha)$, by replacing the condition (1.2) by

$$\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}>\alpha\cos\lambda, z\in U. \tag{1.3}$$

Mogra and Ahuja coined the notion of λ-spirallike functions of order α and type β in [7] by further extending the class of λ-spirallike functions of order α as follows:

A function analytic in U is said to be λ*-spirallike of order* α and type β, if and only if the inequlity

$$\left|\frac{zf'(z)/f(z)}{2\beta\left(zf'(z)/f(z)-1+(1-\alpha)e^{i\lambda}\cos\lambda\right)-\left(zf'(z)/f(z)-1\right)}\right|<1.$$

holds for $0 \le \alpha < 1, 0 < \beta \le 1$ and $-\pi/2 < \lambda < \pi/2$. The class of -spirallike of order α and type β is denoted by $S^{\lambda}(\alpha,\beta)$. Since $S^{\lambda}(\alpha,\beta) \subset S^{\lambda}$, it follows that the functions in $S^{\lambda}(\alpha,\beta)$ are univalent.

Various subclasses of spirallike functions have been introduced and studied extensively by many authors in the literature. A number of results have been proved concerning representation formulae, growth bounds, distortion bounds, extreme points, radii of starllikeness and subordination theorems etc. for functions in these classes. For more information and interesting results on spirallike functions we refer to [1,5,6,7,12,13,14,15, 17,18] and references therein.

In the present paper our aim is to introduce and study a new subclass of spirallike functions denoted by $S^{\lambda}(\alpha,\gamma,\beta)$ which is defined using Ruscheweyh derivative [10]. This subclass will include various subclasses of spirallike and other well known classes of analytic [10]. This subclass will include various subclasses of spirallike and other well known classes of analytic functions as its subclasses. First we give a representation formula for the class $S^{\lambda}(\alpha,\beta,\gamma)$. Next, we shall provide coeffieient bounds and distortion properties for functions in this class. We then prove a subordination theorem for a subclass of $S^{\lambda}(\alpha,\gamma,\beta)$ consisting of Ruscheweyh type analytic functions. Finally, we obtain λ-spiral radius for function in $S^{\lambda}(\alpha,\gamma,\beta)$.

2. The Subclass $S^{\lambda}(\alpha,\gamma,\beta)$. Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in U. We introduce a subclass of A consisting of functions $f(z)$ which satisfy condition

$$\operatorname{Re}\left\{1-\frac{2}{\beta}+\frac{2}{\beta}e^{i\lambda}\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}\right\} > \alpha\cos\lambda, \quad (z \in U), \tag{2.1}$$

for $0 \le \alpha < 1, \beta \ne 0$ and $\gamma > -1$. Here $D^{\gamma}f(z)$ is the Ruscheweyh derivative of f introduced by St. Ruscheweyh [10] defined as

$$D^{\gamma}f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z) = z + \sum_{n=2}^{\infty} a_n A_n(\gamma) z^n. \tag{2.2}$$

here '*' denotes the convolution or Hadamard product of two power series and

$$A_n(\gamma)=\frac{(\gamma+1)(\gamma+2)...(\gamma+n-1)}{(n-1)!}. \tag{2.3}$$

Note that

$$A_n(0)=1, A_n(1)=n \text{ and } A_{n+1}(\gamma)=\frac{1}{\gamma+1}A_n(\gamma+1).$$

We designate the class of functions satisfying condition (2.1) by $S^{\gamma}(\alpha,\beta,\gamma)$. We shall see soon that the subclass $S^{\gamma}(\alpha,\beta,\gamma)$ contains many well known classes of univalent functions studied by several mathematicians in the literature as well as new subclasses of univalent starlike, spirallike and convex functions. Since $S^{\gamma}(\alpha,\beta,\gamma)\subset S^{\lambda}$, it follows that the functions in $S^{\gamma}(\alpha,\beta,\gamma)$ are univalent.

In particular, for $\lambda=\alpha=0, \beta=2, \gamma=0$, then we get $S^0(0,2,0)=S^*$, the class of starlike functions and for $\lambda=0, \beta=2, \gamma=0$, we get $S^0(\alpha,2,0)=S^*(\alpha)$ the subclass of starlike functions of order α.

If we take $\lambda=0, \beta=1, \gamma=1$ then we get the subclass $S^0(\alpha,1,1)=K(\alpha)$ of convex functions of order α. For parametric values $\lambda=0, \beta=\gamma=1$ and $\alpha=1$, then we get the class of convex functions K.

Furthermore taking $\beta=2, \gamma=0$, we obtain the class of λ-spirallike functions of order α with $|\lambda|<\pi/2$.

3. A representation for $S^{\lambda}(\alpha,\beta,\gamma)$. Let A_{ϕ} denote the subclass of functions $\phi\in A$ which are analytic in U and satisfy $|\phi(z)|\le 1$, for all $z\in U$. We first prove the following lemma:

Lemma 3.1. If a function $h(z)=1+\sum_{n=2}^{\infty}c_n z^n$, analytic in U satisfy the condition

$$\left|\frac{\frac{2}{\beta}h(z)-1+e^{-i\lambda}(1-2/\beta)}{\frac{2}{\beta}h(z)-1+e^{-i\lambda}(1-2/\beta+2(1-\alpha)\cos\lambda)}\right|<1 \tag{3.1}$$

for some $0 \le \alpha < 1, \beta \ne 0, \lambda \in (-\pi/2, \pi/2), z \in U$ then

$$h(z) = \frac{1 + e^{-i\lambda}(1 - 2/\beta) + \{(1 - 2/\beta + 2(1-\alpha)\cos\lambda) z\phi(z)\}}{(2/\beta)(1 + z\phi(z))} \tag{3.2}$$

for some $\phi \in A_\phi$. Conversely, any function $h(z)$ given by (3.2) for some $\phi \in A_\phi$ is analytic in U and satisfy (3.1) for all $z \in U$.

Proof. The first part of the Lemma is obtained by an application of Schwartz's Lemma.

To Prove the converse part we see that the function

$$w(z) = \frac{1 + e^{-i\lambda}\left(1 - 2/\beta + \{1 - 2/\beta + 2(1-\alpha)\cos\lambda\}\right) z}{(2/\beta)(1+z)}$$

maps $|z| < 1$ onto the disk

$$\left| \frac{1 - 2/\beta w(z) + e^{-i\lambda}(1 - 2/\beta)}{(2/\beta) w(z) - 1 + e^{-i\lambda}(1 - 2/\beta) + 2(1-\alpha)\cos\lambda} \right|$$

in the w-plane. The conclusion of the Lemma now follows.

We now give a representation formula for functions in the class $S^\lambda(\alpha, \beta, \gamma)$ for a particular choice of the parameter γ. We shall use Lemma 3.1 to prove our theorem.

Theorem 3.1 A function $f(z) \in A$ is in the class $S^\lambda(\alpha, \beta, n)$ if and only if

$$f^{(n)}(z) = \left[z \exp\left\{ e^{-i\lambda}(1 - 2/\beta) \int_0^z \frac{du}{u(1 + u\phi(u))} - e^{-i\lambda}(1 - 2/\beta + 2(1-\alpha)\cos\lambda) \int_0^z \frac{\phi(u)du}{(1 + u\phi(u))} \right\} \right]^{(n+1)} \tag{3.3}$$

$n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for some $\phi \in A_0$.

Proof. Suppose that $f(z) \in S^\lambda(\alpha, \beta, \gamma)$. Let us define

$$g(z) = \frac{D^{n+1} f(z)}{D^n f(z)}, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Note that $g(z)$ satisfies hypothesis of the Lemma 3.1, therefore we can write

$$g(z) = \frac{1 + e^{-i\lambda}\left(1 - 2/\beta + \{1 - e^{-i\lambda}(1 - 2/\beta + 2(1-\alpha)\cos\lambda) z\phi(z)\}\right)}{(1/\beta)(1 + z\phi(z))}, \tag{3.4}$$

for some $\phi \in A_\phi$. From this we obtain

$$\frac{1}{z}g(z)-\frac{1}{z}=\frac{1}{z}\left[\frac{e^{-i\lambda}(1-2/\beta)}{(2/\beta)(1+z\phi(z))}\frac{e^{-i\lambda}(1-2/\beta+2(1-\alpha)\cos\lambda)z\phi(z)}{2/3(1+z\phi(z))}\right]. \tag{3.5}$$

In his paper Ruscheweyh [10] proved that

$$D^n f(z)=\frac{z\left(z^{n-1}f(z)\right)^{(n)}}{n!}, n\in\mathbb{N}_0=\{0,2,\ldots\}.$$

$D^n f(z)$ is known as the n^{th} Ruscheweyh derivative of $f(z)$ (see Al-Amiri [2]).

Using this we get

$$\frac{1}{z}g(z)-\frac{1}{z}=\frac{1}{(n+1)}\frac{f^{(n+1)}(z)}{f^{(n)}(z)}-\frac{1}{z}\ .$$

Thus (3.5) yields

$$\frac{1}{(n+1)}\frac{f^{(n+1)}(z)}{f^{(n)}(z)}-\frac{1}{z}=\frac{1}{z}\left[\frac{e^{-i\lambda}(1-2/\beta)}{(2/\beta)z(1+z\phi(z))}\frac{e^{-i\lambda}(1-2/\beta+2(1-\alpha)\cos\lambda)z\phi(z)}{2/3(1+z\phi(z))}\right].$$

Integrating from 0 to z followed by exponentiation, we get

$$f^{(n)}(z)=\left[z\exp\left\{e^{-i\lambda}(1-2/\beta)\int_0^z\frac{du}{u(1+u\gamma\phi(n))}\right.\right.$$

$$\left.\left.-e^{-i\lambda}(1-2/\beta+2(1-\alpha)\cos\lambda)\int_0^z\frac{\phi(u)du}{u(1+u\phi(n))}\right\}\right]^{(n+1)}.$$

Conversely if (3.3) holds, then

$$\frac{\left(f^{(n)}(z)\right)^{n+1}}{z}=\exp\left\{e^{-i\lambda}(1-2/\beta)\right\}\int_0^x\frac{du}{u(1+u\phi(n))}$$

$$\left.-e^{i\lambda}(1-2/\beta+2(1-\alpha)\cos\lambda)\int_0^z\frac{\phi(u)du}{(1-u\phi(u))}\right\}.$$

Taking log on both sides and using (3.5), we get

$$\frac{d^{n+1}f(z)}{D^n f(z)}=\frac{1}{z}\left[\frac{e^{-i\lambda}(1-2/\beta)-e^{-i\lambda}(1-2/\beta+2(1-\alpha)\cos\lambda)z\phi(z)}{(2/\beta)(1+z\phi(z))}\right].$$

Now the result follows from the converse part of the Lemma 3.1

Remark 3.1 Choosing $\beta = 2, n = 0$ in (3.3), we obtain a representation formula for λ-sparallike functions of order α determined by Liebra [6].

4. A Sufficient Condition. We now establish a sufficient condition for a function to be in the class $S^{\lambda}(\alpha,\beta,\gamma)$. For this purpose we first prove the following Lemma:

Lemma 4.1 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies

$$\left|\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)} - 1\right| < 1-\delta,$$

for $\theta \le \delta < 1$ and $z \in U$.

Then $f(z) \in S^{\lambda}(\alpha,\beta,\gamma)$ for $|\lambda| \le \cos^{-1}\left(\frac{(2/\beta)(2-\delta)-1}{2/\beta-\alpha}\right)$.

Proof. We may write

$$\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)} - 1 = (1-\delta)w(z), \text{ where } |w(z)| < 1, z \in U.$$

Thus, we have

$$\begin{aligned}
\operatorname{Re}\left\{1 - \frac{2}{\beta} + \frac{2}{\beta}e^{i\lambda}\frac{D^{\lambda+1}f(z)}{D^{\gamma}f(z)}\right\} &= \operatorname{Re}\left\{1 - 2/\beta + (2/\beta)e^{i\lambda}\left(1 + (1-\delta)w(z)\right)\right\} \\
&= 1 - 2/\beta + (2/\beta)\cos\lambda + (2/\beta)\operatorname{Re}\left\{e^{i\lambda}\left(1+(1-\delta)w(z)\right)\right\} \\
&\ge 1 - 2/\beta + (2/\beta)\cos\lambda - (2/\beta)(1-\delta)|w(z)| \\
&> 1 - 2/\beta + (2/\beta)\cos\lambda - (2/\beta)(1-\delta) \\
&= 1 + 2/\beta\cos\lambda - (2/\beta)(2-\delta) \\
&\ge \alpha\cos\lambda,
\end{aligned}$$

provided $|\lambda| < \cos^{-1}\left(\frac{(2/\beta)(2-\delta)-1}{2/\beta-\alpha}\right)$ and this completes the proof.

Lemma 4.2 If $\left|\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)} - 1\right| < (2/\beta-\alpha)\cos\lambda$, then $f(z) \in S^{\lambda}(\alpha,\beta,\gamma)$.

Proof. Taking $\delta = s - (\beta/2)\left(1 + ((2/\beta)-\alpha)\cos\lambda\right)$ in previous Lemma, the result follows.

Theorem 4.1 Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$. If

$$\sum_{n=2}^{\infty}\left\{\left((2/\beta)-\alpha\right)+\frac{n-1}{\gamma+1}\sec\lambda\right\}A_n(\gamma)|\alpha_n|<\left((2/\beta)-\alpha\right), \tag{4.1}$$

then $f(z)\in S^{\lambda}(\alpha,\beta,\gamma)$.

Proof. By Lemma 4.2 it is sufficient to show that

$$\left|\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}-1\right|<\left((2/\beta)-\alpha\right)\cos\lambda. \tag{4.2}$$

Note that

$$D^{\gamma+1}f(z)-D^{\gamma}f(z)=\sum_{n=2}^{\infty}a_n\left(A_n(\gamma)-A_{n+1}(\gamma)\right)x^n$$

$$=\sum_{n=2}^{\infty}a_n\left(\frac{n-1}{\gamma+1}\right)A_n(\gamma)z^n.$$

Now consider

$$\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}-1=\frac{\sum_{n=2}^{\infty}a_n\left(\frac{n-1}{\gamma+1}\right)A_n(\gamma)z^n}{z+\sum_{n=2}^{\infty}a_nA_n(\gamma)z^n}.$$

From this we obtain

$$\left|\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}-1\right|=\left|\frac{\sum_{n=2}^{\infty}a_n\left(\frac{n-1}{\gamma+1}\right)A_n(\gamma)z^n}{z+\sum_{n=2}^{\infty}a_nA_n(\gamma)z^n}\right|$$

$$\leq\frac{\sum_{n=2}^{\infty}a_n\left(\frac{n-1}{\gamma+1}\right)A_n(\gamma)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}A_n(\gamma)|a_n||z|^{n-1}}$$

$$\leq\frac{\sum_{n=2}^{\infty}\left(\frac{n-1}{\gamma+1}\right)A_n(\gamma)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}A_n(\gamma)|a_n|}.$$

This is bounded by $(2/\beta-\alpha)\cos\lambda$ if

$$\sum_{n=2}^{\infty}\left(\frac{n-1}{\gamma+1}\right)A_n(\gamma)|a_n| \le (2/\beta-\alpha)\cos\lambda\left(1-\sum_{n=2}^{\infty}A_n(\gamma)|a_n|\right)$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left\{\left(2/\beta-\alpha+(n-1)/(\gamma+1)\sec\lambda\right)A_n(\gamma)\right\}|a_n| < (2/\beta-a).$$

This completes the proof of the theorem.

Taking $\gamma=0$ in (4.1), we get

Corollary 4.1 Let $f(z)\in A$. if $\sum_{n=2}^{\infty}\{2/\beta-\alpha+(n-1)\sec\lambda\}|a_n| < (2/\beta-\alpha)$ then $f(z)\in S^{\lambda}(\alpha,\beta,0)$.

The follwoing corollary is due to Kwon and Owa [5] which corresponds to $\beta=2,\gamma=0$ in (1.1).

Corollary 4.2 Let $f(z)\in A$ if $\sum_{n=2}^{\infty}\{(n+1)+(1-\alpha)\cos\lambda\}|a_n| < 1-\alpha$ then $f(z)\in S^{\lambda}(\alpha,2,0)=S^{\lambda}(\alpha)$.

Remark 4.1. For parametric value $\lambda=0,\beta=2$ and $\gamma=0$ we obtain a sufficient condition for *f(z)* to be starlike of order α proved by Silvernman [13].

Now we shall prove the necessary coefficient condition for functions in the class $S^{\lambda}(\alpha,\beta,\gamma)$.

Theorem 4.2 Let $f(z)\in S^{\lambda}(\alpha,\beta,\gamma)$ then

$$|a_2| \le \beta(1-\alpha\cos\lambda) \tag{4.3}$$

and

$$|a_n| \le \frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{(n-1)A_n(\gamma)}\prod_{j=1}^{n-2}\left(1+\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{j}\right). \tag{4.4}$$

Proof. Since $f(z)\in S^{\lambda}(\alpha,\beta,\gamma)$, therefore

$$\operatorname{Re}\left\{1-\frac{2}{\beta}+\frac{2}{\beta}e^{i\lambda}\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}\right\} > \alpha\cos\lambda, (z\in U).$$

Define the function $q(z)$ by

$$q(z)=\frac{1+(2/\beta)e^{i\lambda}\dfrac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}-\left(\alpha\cos\lambda+(2/\beta)e^{i\lambda}\right)}{1-\alpha\cos\lambda},(z\in U).$$

Then $q(z)$ is analytic in U and satisfies $q(0)=1,\mathrm{Re}(q(z))>0$.

Let us assume that

$$q(z)=1+q_1z+q_2z+\ldots.$$

Then, we have

$$1+(2/\beta)e^{i\lambda}\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}=(1-\alpha\cos\lambda)\left\{1+\sum_{n=1}^{\infty}q_nz^n\right\}+\alpha\cos\lambda+(2/\beta)e^{i\lambda}$$

$$=1+(1-\alpha\cos\lambda)\sum_{n=1}^{\infty}q_nz^n+(2/\beta)e^{i\lambda},$$

which on simplification gives

$$(2/\beta)e^{i\lambda}\left(D^{\gamma+1}f(z)-D^{\gamma}f(z)\right)=D^{\gamma}f(z)\left\{(1-\alpha\cos\lambda)\sum_{n=1}^{\infty}q_nz^n\right\}.$$

From this, we obtain

$$\frac{2}{\beta}e^{i\lambda}\frac{n-1}{\gamma+1}A_n(\gamma)a_n=(1-a\cos\lambda)\left\{q_1A_{n-1}(\gamma)a_{n-1}+q_2A_{n-2}(\gamma)a_{n-2}+\ldots+q_{n-2}A_2(\gamma)a_2+q_{n-1}\right\}$$

Now applying the co-efficient estimates $|q_n|\le 2$ for Caratheodory function [3], we see that

$$|a_n|\le\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{(n-1)A_n(\gamma)}\left\{1+A_2(\gamma)|a_2|+A_3(\gamma)|a_3|+\ldots+A_{n-2}(\gamma)|a_{n-2}|+A_{n-1}(\gamma)|a_{n-1}|\right\}.$$

For $n=2$, we get

$$|a_2|\le\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{(2-1)A_2(\gamma)}=\beta(1-\alpha\cos\lambda).\tag{4.5}$$

This proves (4.3) for $n=2$.

For $n=3$, we get

$$|a_3|\le\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{2A_3(\gamma)}\left\{1+A_2(\gamma)|a_2|\right\}\tag{4.6}$$

$$=\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{2A_3(\gamma)}\left\{1+\beta(1-\alpha\cos\lambda)(\gamma+1)\right\}.$$

Thus, (4.4) holds for $n=3$.

Let us assume that (4.4) is true for $n=k$, then

$$|a_{k+1}| \le \frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{kA_{k+1}(\gamma)}\left\{(1+\beta(1-\alpha\cos\lambda)(\gamma+1)) + \frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{A_2(\gamma)}\right.$$

$$\left.(1+\beta(1-\alpha\cos\lambda)(\gamma+1)) + \dots + \frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{(k-1)!A_k(\gamma)}\prod_{j=1}^{k-2}\left(1+\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{j}\right)\right.$$

$$= \frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{kA_{k+1}(\gamma)}\prod_{j=1}^{k-1}\left(1+\frac{\beta(1-\alpha\cos\lambda)(\gamma+1)}{j}\right).$$

Therefore, the result is true for $n = k+1$. Hence using mathematical induction (4.4) holds for all positive integer $n > 3$.

For parametric value , we get

Corollary 4.3. If $f(z) \in S^{\lambda}(\alpha,2,0) = S^{\lambda}(\alpha)$. then

$$|a_2| \le 2(1-\alpha\cos\lambda) \text{ and}$$

$$|a_n| \le \frac{2(1-\alpha\cos\lambda)}{(n-1)}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\alpha\cos\lambda)}{j}\right), n \ge 3.$$

Taking $\beta = 2, \gamma = 0, \lambda = 0$ in Theorem 4.2, we get a result of Robertson [8].

Corollary 4.4. If $f(z) \in S^{(0)}(\alpha,2,0) = S^{*}$ then

$$|a_2| \le 2(1-\alpha) \text{ and } |a_n| \le \prod_{j=2}^{n-1}\frac{(j-2\alpha)}{(n-1)!}, n \ge 3.$$

5. The η-spiral radius. Let S be the family of all normalized functions which are analytic and univalent in U. Following Libera [6]. if $f \in S$ and $|\eta| < \pi/2$, then η-spiral radius of f is defined by

$$\eta - s.r.\{f\} = \sup\left\{r : \operatorname{Re}\left(e^{i\eta}\frac{zf'(z)}{f(z)}\right) > 0, |z| < r\right\}$$

and if $E \subseteq S$. then the η-sprial radius of E is defined by

$$\eta - s.r.E = \inf_{f \in E}\{\eta - s.r.\{f\}\}. \tag{5.1}$$

We shall now determine the η-spiral radius of the class $S^{\lambda}(\alpha,\beta,\gamma)$ for γ $=1$.

Theorem5.1 $\eta - s.r.S^{\lambda}(\alpha,\beta,1)$ is the smallest positive root r of the equation

$$\{(1-2/\beta+2(1-\alpha)\cos\lambda)\cos(\eta-\lambda)-\cos\eta\}r^2$$

$$-\left(1-\beta/2+2(1-\alpha)\cos\lambda\right)r+\cos\eta\geq 0.$$

Proof. Let $f(z)\in S^{\lambda}(\alpha,\beta,a)$. Then by Lemma 3.1,

$$\frac{D^{\gamma+1}f(z)}{D^{\gamma}f(z)}=\frac{1+(1-2/\beta)e^{-i\lambda}+\left\{1-\left(1-Z/\beta+z(1-\alpha)\cos\lambda\right)e^{-i\lambda}\right\}w(z)}{1+w(z)},$$

where $w(z)$ satisfies $w(0)=0$ and $|w(z)|<1$.

If $\gamma=0$ and $A(z)=e^{in}\dfrac{zf'(z)}{f(z)}$ then (5.3) may be written as

$$\left(1+w(z)\right)A(z)=e^{i\eta}+(1-Z/\beta)e^{i(\eta-\lambda)}+\left\{e^{i\eta}-\left(1-2/\beta+z(1-\alpha)\cos\lambda\right)e^{i(\eta-\lambda)}\right\}w(z).$$

Therefore.

$$w(z)=\frac{e^{i\eta}+(1-2/\beta)e^{i(\eta-\lambda)}-A(z)}{A(z)-e^{i\eta}+\left(1-2/\beta+2(1-\alpha)\cos\lambda\right)e^{i(n-\lambda)}},(z\in U). \tag{5.4}$$

Now applying Schwartz's Lemma, it follows that $A(z)$ maps the disk $|z|\leq r$ onto a disk $|A(z)-\theta|<R$. were

$$\theta=\frac{e^{i\eta}-\left(1-2/\beta+2(1-\alpha)\cos\lambda\right)e^{i(\eta-\lambda)}r^{2}}{1-r^{2}} \text{ and } R=\frac{\left(1-\beta/2+2(1-\alpha)\cos\lambda\right)r}{1-r^{2}} \tag{5.5}$$

Thus, $\mathrm{Re}\left\{e^{i\eta}\dfrac{zf'(z)}{f(z)}\right\}\geq 0$ if and only if

$$\mathrm{Re}\left\{\frac{e^{i\eta}-\left[e^{i\eta}-\left(1-2/\beta+2(1-\alpha)\cos\lambda\right)e^{i(\eta-\lambda)}\right]r^{2}}{1-r^{2}}\right\}$$

$$\geq\frac{\left(1-\beta/2+2(1-\alpha)\cos\lambda\right)r}{1-r^{2}},$$

which on simplification gives,

$$\left\{\left(1-2/\beta+2(1-\alpha)\cos\lambda\right)\cos(\eta-\lambda)-\cos\eta\right\}r^{2}$$
$$-\left(1-\beta/2+2(1-\alpha)\cos\lambda\right)r+\cos\eta\geq 0. \tag{5.6}$$

This with the aid of (5.1) completes the proof of the theorem.

Remark 5.1 By choosing suitable values of the parameters $\alpha,\beta,\gamma,\lambda$ and η in the previous theorem, we obtain the corresponding results for the several well known subclasses of S.

6. A Subordination Theorem. We know that if $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ are analytic in U, then their Hadamard product denoted by $f*g$ is defined by the power series

$$f*g(z)=\sum_{n=0}^{\infty} a_n b_n z^n. \tag{6.1}$$

Clearly, the funciton $f*g$ is also analytic in U.

We also know that a function $f(z)\in S$ is said to be convex if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0\,(z\in U). \tag{6.2}$$

The class of all convex functions is denoted by K.

Definition 6.1. Let f is analytic in, U,g analytic and univalent in U and $f(0)=g(0)$. Then we say that f is subordinate to g, denoted by the symbol $f(z)\prec g(z)$, in U if $f(U)\subset g(U)$.

Definition 6.2. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if whenever $f(z)=\sum_{n=1}^{\infty} a_n z^n, a_1=1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \text{ in U}. \tag{6.3}$$

We now state a Lemma due to Wilf [16] which we shall need to prove our subordination theorem.

Lemma 6.1 The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\}>0\,(z\in U). \tag{6.4}$$

We are now in a position to state our subordination theorem for the subclass $S^{\lambda}(\alpha,\beta,\gamma)$.

Theorem 6.1 Let $f(z)\in S^{\lambda}(\alpha,\beta,\gamma)$. Then for any $g(z)\in K$, we have

$$\frac{(2/\beta-\alpha)+\sec\lambda}{2\left(2(2/\beta-\alpha)+\sec\lambda\right)}\left(D^{\gamma}f*g\right)(z)\prec g(z) \text{ for } z\in U. \tag{6.5}$$

Here $D^{\gamma}f$ is the Ruschewehy derivative of f as defined in (2.2). In particular

$$\mathrm{Re}\left(D^{\gamma}f(z)\right)>-\frac{2(2/\beta-\alpha)+\sec\lambda}{(2/\beta-\alpha)+\sec\lambda}(z\in U). \tag{6.6}$$

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in $S^{\lambda}(\alpha,\beta,\gamma)$ and let $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ be in K, Then

$$\frac{(2/\beta-\alpha)}{2\left(2(2/\beta-\alpha)+\sec\lambda\right)}\left(D^{\gamma}f*g\right)(z)=\frac{(2/\beta-\alpha)+\sec\lambda}{2\left(2(2/\beta-\alpha)+\sec\lambda\right)}\left(z+\sum_{n=2}^{\infty}a_nA_n(\gamma)b_nz^n\right). \tag{6.7}$$

Using definition 6.2, it is enough to show that the sequence

$$\left\{\frac{\{(2/\beta-\alpha)+\sec\lambda\}a_nA_n(\gamma)}{2\left(2(2/\beta-\alpha)+\sec\lambda\right)}\right\}_{n=1}^{\infty},$$

is a subordinating factor sequence with $a_1=1$. Now using Lemma 6.1 this will the case if and only if

$$\mathrm{Re}\left\{1+2\sum_{n=1}^{\infty}\frac{(2/\beta-\alpha)+\sec\lambda}{2\left(2(2/\beta-\alpha)+\sec\lambda\right)}a_nA_n(\gamma)z^n\right\}>0(z\in U).$$

Let us consider

$$\mathrm{Re}\left\{1+\sum_{n=1}^{\infty}\frac{(2/\beta-\alpha)+\sec\lambda}{2\left(2(2/\beta-\alpha)+\sec\lambda\right)}a_nA_n(\gamma)z^n\right\}$$

$$=\mathrm{Re}\left\{1+\frac{(2/\beta-\alpha)+\sec\lambda}{2(2/\beta-\alpha)+\sec\lambda}z+\frac{(2/\beta-\alpha)}{\left(2(2/\beta-\alpha)+\sec\lambda\right)}\sum_{n=2}^{\infty}\left(1+\frac{\sec\lambda}{(2/\beta-\alpha)}\right)a_nA_n(\gamma)z^n\right\}$$

$$\geq 1-\frac{(2/\beta-\alpha)+\sec\lambda}{2(2/\beta-\alpha)+\sec\lambda}r-\frac{(2/\beta-\alpha)}{2(2/\beta-\alpha)+\sec\lambda}\sum_{n=2}^{\infty}\left(1+\frac{(n-1)\sec\lambda}{(\gamma+1)(2/\beta-\alpha)}\right)|a_n|A_n(\gamma)r^n$$

$(|z|=r)$

$$\geq 1-\frac{(2/\beta-\alpha)+\sec\lambda}{2(2/\beta-\alpha)+\sec\lambda}r-\frac{(2/\beta-\alpha)}{2(2/\beta-\alpha)+\sec\lambda}r \text{ (using 1.1)}$$

$$=1-r$$

$> 0.$

This prove that part of the theorem. The assertion that $\operatorname{Re}\left(D^{\gamma} f(z)\right) >$ $\dfrac{2(2/\beta-\alpha)+\sec\lambda}{(2/\beta-\alpha)+\sec\lambda}$ for $f(z) \in S^{\lambda}(\alpha,\beta,\gamma)$ follows by taking $g(z) = \dfrac{z}{(1-z)^{\gamma+1}}$ in (6.5).

For $\beta = 2, \gamma = 0, \lambda = 0$, we obtain the following corollary for starlike functions

Corollary 6.1 Let $f(z)$ is regular in U and stisfies the condition

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \leq 1,$$

then for every function $g \in K$, we have

$$\sum_{n=2}^{\infty} \frac{2-\alpha}{2(3-2\alpha)} (f * g)(z) \prec g(z) \leq 1.$$

Remark 6.1 For different values of the parameters λ, α, β and γ in Theorem 6.1 we get carre sponding results of [5,7,14] and references therein.

Remark 6.2 By a proper choice of the parameters λ, α, β and γ in Theorem 5.1 we obtain η-spiral radius and radius of starlikeness of the classes studied by [6,7,9] an others.

Remark 6.3 For suitable values of the parameters λ, α, β and γ in all other results of the paper too get results of [5,6,7,9,14] and references therein along with several new results.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

ON CERTAIN UNIFIED FRACTIONAL DERIVATIVES PERTAINING TO PRODUCT OF *H*-FUNCTION AND GENERALIZED MULTIVARIABLE POLYNOMIALS

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(Received : October 25, 2015; Revised : November 14, 2015)

ABSTRACT

In the present paper, we establish two unified fractional derivative formulas of Saigo and Maeda type, pertaining to product of $\bar{H}$-function and Srivastava's generalized multivariable polynomials. Our main results are in the form of two theorems and provide unification and extension to many known results obtained earlier in the literature. On account of being general in nature our main findings also yield a large number of known and (presumably) new results involving, for example, Saigo fractional calculus operators, several special functions like *H*-function. For the sake of illustration, we also record some special cases.

2010 Mathematics Subject Classification : Primary 26E33, 33E20, 33C45; Secondary 33C60, 33D70.

Keywords: Generalized fractional calculus operators, generalized class of polynomials, $\bar{H}$-function.

1 Introduction. In fractional calculus we investigate integrals and derivatives of arbitrary orders. The fractional derivative operators involving various special functions have found significant importance and applications in various fields of applied mathematics. Many research workers have studied certain properties, applications and different extensions of various hypergeometric operators of fractional differentiations.

Let $\alpha,\alpha',\beta,\beta',\gamma \in C$ with $\Re(\gamma)>0$ and $x \in R_+$, then the generalized fractional differential operators involving the Appell function F_3 in the kernel are defined as follows [8]:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\left(I_{0+}^{-\alpha',-\alpha-\beta',-\beta,-\gamma} f\right)(x) \tag{1.1}$$

$$=\left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f\right)(x) \quad \left(\Re(\gamma)>0; n=\left[\Re(\gamma)\right]+1\right) \tag{1.2}$$

$$=\frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \left(x^{\alpha'}\right)$$

$$\int_0^x (x-t)^{n-\gamma-1} t^{\alpha} F_3\left(-\alpha',-\alpha,n-\beta',-\beta,n-\gamma;1-\frac{1}{x},1-\frac{x}{t}\right) f(t)\,dt \tag{1.3}$$

and

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{1.4}$$

$$=\left(-\frac{d}{dx}\right)^n \left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f\right)(x)\left(\Re(\gamma)>0; n=\left[\Re(\gamma)\right]+1\right) \tag{1.5}$$

$$=\frac{1}{\Gamma(n-\gamma)}\left(-\frac{d}{dx}\right)^n \left(x^{\alpha}\right)\int_x^{\infty} (t-x)^{n-\gamma-1} t^{\alpha'}$$

$$F_3\left(-\alpha',-\alpha,-\beta',n-\beta,n-\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt\ . \tag{1.6}$$

where F_3 is one of the Appell series defined by [10]

$$F_3(a,a',b,b';c;x,y)=\sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} \quad \left(\max\{|x|,|y|\}<1\right) \tag{1.7}$$

and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in C$) by [11]

$$(\lambda)_n=\begin{cases}1 & (n=0),\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in N)\end{cases}$$

$$=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\left(\lambda \in C/Z_0^-\right)\ . \tag{1.8}$$

These operators reduce to the Saigo derivative operators as follows [7,8]:

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x)=\left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x) \quad \left(\Re(\gamma)\right)>0 \tag{1.9}$$

and

$$\left(D_{0-}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{0-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x) \quad \left(\Re(\gamma)\right)>0 \ . \tag{1.10}$$

Furthermore we also have [8]

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=\Gamma\begin{bmatrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta,\rho+\beta'\end{bmatrix}x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\left(\Re(\gamma)>0,\left(\Re(\rho)>\max\left\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\right\}\right)\right) \tag{1.11}$$

and

$$I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=\Gamma\begin{bmatrix}1+\alpha+\alpha'-\gamma-\rho,1+\alpha+\beta'-\gamma-\rho,1-\beta-\rho \\ 1-\rho,1+\alpha+\alpha'+\beta'-\gamma-\rho,1+\alpha-\beta-\rho\end{bmatrix}x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\left(\Re(\gamma)>0,\left(\Re(\rho)<1+\min\left\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\right\}\right)\right) \tag{1.12}$$

where $\Gamma[\ldots]$ represents the ratio of the product of Gamma functions, for example,

$$\Gamma\begin{bmatrix}\alpha,\beta,\gamma \\ a,b,c\end{bmatrix}=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(a)\Gamma(b)\Gamma(c)}.$$

The $\bar{H}$-function appearing in this paper is defined and represented as follows [2].

$$\bar{H}_{P,Q}^{M,N}[z]=\bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_1)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_{-i\infty}^{+i\infty}\phi(\xi)z^{\xi}d\xi, \tag{1.13}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}, \tag{1.14}$$

which contains fractional powers of some of the gamma functions. Here, and throughout the paper $a_j(j=1,\ldots,P)$, and $b_j(j=1,\ldots,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,\ldots,P),\beta_j\geq 0(j=1,\ldots,Q)$ (not all zero simultaneously) and the exponents $A_j(j=1,\ldots,N)$ and $B_j(j=M+1,\ldots,Q)$ can take non-integer values.

Also $\left|\arg z\right| < \frac{1}{2}\theta\pi$ (1.15)

where

$$\theta = \sum_{j=1}^{M}\left|\beta_j\right| + \sum_{j=1}^{N}\left|A_j\alpha_j\right| - \sum_{j=N+1}^{Q}\left|\alpha_j\right| > 0, \tag{1.16}$$

The behaviour of the $\bar{H}$-function for small values of follows easily from a result recently given by Rathie [6, p. 306, eq. (6.9)], we have

$$\bar{H}_{P,Q}^{M,N}\left[z\right] = o\left(\left|z\right|^{\alpha}\right), \alpha = \underset{1\le j\le M}{Min.}\left\{\mathrm{Re}\left(b_j/\beta_j\right)\right\} \text{ for small } \left|z\right|. \tag{1.17}$$

For more details one may refer to [1,5].

For the present study, we use the Srivastava's generalized multivariable polynomials ([9],P.185,eq.(7)) defined as:

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[x_1,\dots,x_r\right] =$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{\left(-n_1\right)_{m_1k_1}}{k_1!} \cdots \frac{\left(-n_r\right)_{m_rk_r}}{k_r!} A[n_1,k_1;\dots;n_r,k_r]x_1^{k_1}\dots,x_r^{k_r} \tag{1.18}$$

where n_i=0,1,2,...(I=1,...,r), $m_1,\dots,m_r$ are arbitrary positive integers and the coefficients $A[n_1,k_1;\dots;n_r,k_r]$ are arbitrary constants, real or complex.

2. Fractional derivative formulae. In this section we establish two theorems containing fractional derivative formulae involving $\bar{H}$-function and Srivastava's generalized multivariable polynomials.

Theorem 1. Let $\alpha,\alpha',\beta,\beta',\gamma,z,\rho \in C, \Re(\gamma) > 0, \mu > 0, \lambda_j \in \Re_+ (j=1,\dots,r)$, and

$$\Re(\rho) + \mu \min_{1\le j\le m}\frac{\Re(b_j)}{\beta_j} + \max\left\{0, \Re(\alpha-\beta), \Re(\alpha'+\beta'+\alpha-\gamma)\right\}$$

and the conditions (1.15) to (1.17) are satisfied, then we get

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}(y_1t^{\lambda_1},\dots,y_rt^{\lambda_r})\bar{H}_{P,Q}^{M,N}\left[zt^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{\left(-n_1\right)_{m_1k_1}}{k_1!} \cdots \frac{\left(-n_r\right)_{m_rk_r}}{k_r!} A[n_1,k_1;\dots;n_r,k_r]y_1^{k_1},\dots,y_1^{k_r}\ x^{\sum_{j=1}^{r}\lambda_jk_j}$$

$$\bar{H}^{M,N+3}_{P+3,Q+3}\left[zx^{\mu}\left|\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_j k_j;\mu;1),(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\gamma-\alpha-\alpha'-\beta';\mu;1),\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\gamma-\alpha-\alpha';\mu;1),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\beta-\alpha;\mu;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\gamma-\alpha-\beta';\mu;1),(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\beta;\mu;1)\end{matrix}\right]. \qquad (2.1)$$

Proof. We can write

$$\text{L.H.S. of (2.1)} = \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y^{k_1},\ldots,y^{k_r}$$

$$\frac{1}{2\pi\omega}\left\{\int_L\phi(\xi)z^{\xi}\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}t^{\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-1}\right)(x)d\xi\right\}.$$

$$\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y^{k_1},\ldots,y^{k_r}$$

$$\frac{1}{2\pi\omega}\left\{\int_L\phi(\xi)z^{\xi}\left(I^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}_{0+}t^{\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-1}\right)(x)d\xi\right\} \quad \text{(By an appeal to (1.1)}$$

$$=x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y^{k_1}\ldots y^{k_r}\ x^{\sum_{j=1}^{r}\lambda_j k_j}$$

$$\frac{1}{2\pi\omega}\int_L\phi(\xi)(zx^{\mu})^{\xi}\Gamma\left[\begin{matrix}\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi,\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-\gamma+\alpha+\alpha'+\beta',\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-\beta+\alpha\\\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-\gamma+\alpha+\alpha',\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-\gamma+\alpha+\beta',\rho+\sum_{j=1}^{r}\lambda_j k_j+\mu\xi-\beta\end{matrix}\right]$$

$$=x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y^{k_1}\ldots y^{k_r}\ x^{\sum_{j=1}^{r}\lambda_j k_j}$$

$$\frac{1}{2\pi\omega}\int_L (zx^{\mu})^{\xi}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}\frac{\Gamma((\rho+\sum_{j=1}^{r}\lambda_j k_j)+\mu\xi)}{\Gamma((\rho+\sum_{j=1}^{r}\lambda_j k_j-\gamma+\alpha+\alpha')+\mu\xi)}$$

$$\frac{\Gamma((\rho+\sum_{j=1}^{r}\lambda_j k_j-\gamma+\alpha+\alpha'+\beta')+\mu\xi))\Gamma((\rho+\sum_{j=1}^{r}\lambda_j k_j-\beta+\alpha)+\mu\xi))}{\Gamma((\rho+\sum_{j=1}^{r}\lambda_j k_j-\gamma+\alpha+\beta')+\mu\xi)\Gamma((\rho+\sum_{j=1}^{r}\lambda_j k_j-\beta)+\mu\xi)}d\xi.$$

Finally re-interpreting the Mellin-Barnes contour integral in terms of $\bar{H}$-function we arrive at the required result.

In view of the relationship (1.9), we obtain

Corollary 1. Let $\alpha,\beta,\gamma,z,\rho\in C,\Re(\alpha)>0,\mu>0,\lambda_j\in\Re_+(j=1,...,r)$, and

$$\Re(\rho)+\mu\min_{1\le j\le m}\frac{\Re(a_j)}{A_j}+\max\{0,\Re(\beta),\Re(\beta+\alpha+\gamma)\}>0$$

then we have

$$\left\{D_{0+}^{\alpha,\beta,\gamma}\left(t^{\rho-1}S_{n_1,...,n_r}^{m_1,...,m_r}(y_1t^{\lambda_1},...,y_rt^{\lambda_r})\bar{H}_{P,Q}^{M,N}\left[zt^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=x^{\rho+\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;...;n_r,k_r]y_1^{k_1}\cdots y_r^{k_r}\,x^{\sum_{j=1}^{r}\lambda_jk_j}$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[zx^{\mu}\left|\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_jk_j;\mu;1),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j-\gamma-\alpha-\beta;\mu;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sum_{j=1}^{r}\lambda_jk_j-\beta;\mu;1),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j-\gamma;\mu;1)\end{matrix}\right.\right]\quad(2.2)$$

Theorem 2. Let $\alpha,\alpha',\beta,\beta',\gamma,z,\rho\in C,\Re(\gamma)>0,\mu>0,\lambda_j\in\Re_+(j=1,...,r)$, and

$$\Re(\rho)+\mu\max_{1\le j\le m}\frac{\Re(a_j)-1}{A_j}<1+\min\{\Re(-\beta),\Re(\gamma-\alpha-\alpha'-k),\Re(\alpha'-\beta+\gamma)\}$$

and the conditions (1.15) to (1.17) are satisfied, then we get

$$\left\{D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}S_{n_1,...,n_r}^{m_1,...,m_r}(y_1t^{\lambda_1},...,y_rt^{\lambda_r})\bar{H}_{P,Q}^{M,N}\left[zt^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y^{k_1}\ldots y^{k_r}\,x^{\sum_{j=1}^{r}\lambda_jk_j}$$

$$\bar{H}^{M+3,N}_{P+3,Q+3}\left[zx^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(1-\rho-\sum_{j=1}^{r}\lambda_jk_j;\mu),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\alpha'-\beta;\mu),\\ (1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\beta';\mu),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\beta-\alpha';\mu),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\alpha';u\right),(a_j,\alpha_j)_{N+1,P}\\ \left(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha';u\right),(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right]. \tag{2.3}$$

Proof. Again on proceeding similarly as in the above Theorem, we get

$$\text{L.H.S. of (2.3)} = \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y_1^{k_1}\ldots y_r^{k_r}$$

$$\frac{1}{2\pi\omega}\left\{\int_L\phi(\xi)z^{\xi}\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+\sum_{j=1}^{r}\lambda_jk_j+\mu\xi-1}\right)(x)d\xi\right\}$$

Now using (1.4) we obtain

$$\text{L.H.S. of (2.3)} = \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y_1^{k_1}\ldots y_r^{k_r}$$

$$\frac{1}{2\pi\omega}\left\{\int_L\phi(\xi)z^{\xi}\left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}t^{\rho+\sum_{j=1}^{r}\lambda_jk_j+\mu\xi-1}\right)(x)d\xi\right\}$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y_1^{k_1}\ldots,y_r^{k_r}\,x^{\sum_{j=1}^{r}\lambda_jk_j}$$

$$\frac{1}{2\pi\omega}\int_L\phi(\xi)(zx^{\mu})^{\xi}$$

$$\Gamma\left[\begin{matrix}1-\alpha-\alpha'+\gamma-\rho-\sum_{j=1}^{r}\lambda_j k_j-\mu\xi, 1-\alpha'-\beta+\gamma-\rho-\sum_{j=1}^{r}\lambda_j k_j-\mu\xi, 1+\beta'-\rho-\sum_{j=1}^{r}\lambda_j k_j-\mu\xi\\ 1-\rho-\sum_{j=1}^{r}\lambda_j k_j-\mu\xi, 1-\alpha-\alpha'-\beta+\gamma-\rho-\sum_{j=1}^{r}\lambda_j k_j-\mu\xi, 1-\alpha'+\beta'-\rho-\sum_{j=1}^{r}\lambda_j k_j-\mu\xi\end{matrix}\right]$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y_1^{k_1}\ldots,y_r^{k_r}\; x^{\sum_{j=1}^{r}\lambda_j k_j}$$

$$\frac{1}{2\pi\omega}\int_L (zx^{\mu})^{\xi}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}\frac{\Gamma((1-\alpha-\alpha'+\gamma-\rho-\sum_{j=1}^{r}\lambda_j k_j)-\mu\xi)}{\Gamma((1-\rho-\sum_{j=1}^{r}\lambda_j k_j)-\mu\xi)}$$

$$\frac{\Gamma((1-\alpha'-\beta'+\gamma-\rho-\sum_{j=1}^{r}\lambda_j k_j)-\mu\xi)\Gamma((1+\beta'-\rho-\sum_{j=1}^{r}\lambda_j k_j)-\mu\xi)}{\Gamma((1-\alpha-\alpha'-\beta+\gamma-\rho-\sum_{j=1}^{r}\lambda_j k_j)-\mu\xi)\Gamma((1-\alpha'+\beta'-\rho-\sum_{j=1}^{r}\lambda_j k_j)-\mu\xi)}d\xi.$$

Finally re-interpreting the Mellin-Barnes contour integral in terms of $\bar{H}$-function we arrive at the required result.

In view of the relationship (1.10), we obtain

Corollary 2. Let $\alpha, \beta, \gamma, z, \rho \in C, \Re(\alpha)>0, \mu>0, \lambda_j\in\Re_+(j=1,\ldots,r)$ and

$$\Re(\rho)+\mu\max_{1\le j\le n}\left(\frac{\Re(a_j)-1}{A_j}\right)<1+\min\{0,[\Re(\alpha)]-\Re(\beta)-1,\Re(\alpha+\gamma)\}$$

then we have

$$\left\{D_{0-}^{\alpha,\beta,\gamma}\left(t^{\rho-1}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(y_1t^{\lambda_1},\ldots,y_rt^{\lambda_r})\bar{H}_{P,Q}^{M,N}\left[zt^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\rho+\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y_1^{k_1}\ldots y_r^{k_r}\; x^{\sum_{j=1}^{r}\lambda_j k_j}$$

$$\bar{H}_{P+2,Q+2}^{M+2,N}\left[zx^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(1-\rho-\sum_{j=1}^{r}\lambda_j k_j;\mu),(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\gamma-\beta;\mu),(a_j,\alpha_j)_{N+1,P}\\(1-\rho-\sum_{j=1}^{r}\lambda_j k_j-\beta;\mu),(1-\rho-\sum_{j=1}^{r}\lambda_j k_j+\alpha+\gamma;\mu),(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] \quad (2.4)$$

3. Special Cases.

(i) Taking $A_j = B_j = 1$ in Theorem 1, we get the results in terms of well known Fox's H-function [4].

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(y_1t^{\lambda_1},\ldots,y_rt^{\lambda_r})H_{P,Q}^{M,N}\left[zt^{\mu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\ldots;n_r,k_r]y_1^{k_1}\ldots y_r^{k_r}\,x^{\sum_{j=1}^{r}\lambda_jk_j}$$

$$H_{P+3,Q+3}^{M,N+3}\left[zx^{\mu}\left|\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_jk_j;\mu),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\alpha'-\beta';\mu),\\(b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q},(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\alpha';\mu),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\beta-\alpha;\mu),(a_j,\alpha_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\beta';\mu),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\beta;\mu)\end{matrix}\right]. \tag{3.1}$$

(ii) Setting $A(n_1,k_1;\ldots;n_r,k_r) = \prod_{j=1}^{r}A(n_j,k_j)$ in the Theorem 1, we get

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\prod_{j=1}^{r}S_{n_j}^{m_j}\left[y_jt^{\lambda_j}\right]\bar{H}_{P,Q}^{M,N}\left[zt^{\mu}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}\prod_{j=1}^{r}A(n_j,k_j)\prod_{j=1}^{r}y_j^{k_j}\,x^{\sum_{j=1}^{r}\lambda_jk_j}$$

$$\bar{H}_{P+3,Q+3}^{M,N+3}\left[zx^{\mu}\left|\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_jk_j;\mu;1),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\alpha'-\beta';\mu;1),\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\alpha';\mu;1),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\beta-\alpha;\mu;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\gamma-\alpha-\beta';\mu;1),(1-\rho-\sum_{j=1}^{r}\lambda_jk_j+\beta;\mu;1)\end{matrix}\right] \tag{3.2}$$

(iii) Taking $A[v_1,k_1;...;v_r,k_r]=\frac{(\beta_1)_{k_1\phi_1+...+k_r\phi_r}}{(v_1)_{k_1\psi_1+...+k_r\psi_r}}$ in Theorem 1, the Srivastava's polynomials occurring therein the LHS of Theorem 1, then it reduces to the general class of multivariable polynomials given by Srivastava and Garg [12] and we obtain the result due to Choi and Kumar [3] .

The theorem 2 also yields a large number of known and new results on suitable specification of parameters involved therein. However, for the space constrains, we do not record them here explicitly.

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Jñānābha , Vol. 45, 2015

(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

ANALYTIC AND NUMERICAL ANALYSIS OF AN EPIDEMIC MODEL WITH VACCINATION

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(*Received : November* 25, 2015)

ABSTRACT

In this paper, we investigate the spread of epidemic like measles, chicken pox etc. in a finite population with susceptible-infected (mild)-infected (severe)-removed (SI_mI_sR) population. This investigation incorporates population- dependent death rate and provision of vaccination. The analysis is carried out to find out the equilibrium points. The four equilibrium points are obtained. We also decide the vaccination rate to eradicate the disease. It is established that the epidemic dies out if the basic reproduction rate is less than 1 and it becomes endemic if the basic reproduction rate is greater than 1. The effects of vaccination, mild infection rate and mild to severity infection rate on the spread of epidemic are evaluated numerically. The results of the computational simulation reveal that vaccination strategy to eradicate the disease is most successful.

2010 Mathematics Subject Classification: Primary 92D30, 92C60; Secondary 92D25

Keywords : Epidemiology, Stability Analysis, Equilibrium points, Population-dependent death rate, Perturbation.

1. Introduction. The study of epidemics has a long history with a vast variety of models and explanations for the spread and cause of epidemic outbreak. Many mathematical models on epidemics have been developed for the transmission dynamics within the various types of populations. A trait can be a disease such as measles, *HIV*, tuberculosis, malaria etc. One of the earliest mathematical models on epidemics was formulated by Kermack and McKendrick [1] to predict the behavior of the epidemics. The first attempt to formulate an *SIR* model with births and deaths to describe measles was given by Soper [21], who considered constant death rate and constant birth rate. Biologically, this model was not compatible to the real situation. A model that includes births in the

susceptible class proportional to total population size and a death rate in each class proportional to the number of members in the class was developed by Kermack and McKendrick [16]. In most of the infectious diseases there is an exposed period after the transmission of infection from susceptible to potential infected members before the time when these potential infected can retransmit infection.

The models which are used for viral diseases as measles, chicken pox, were formulated (cf. Dietz [7], Anderson and May [1]). Epidemiological models focus on the transmission dynamics of trait or traits transmitted from individual to individual, from population to population, from community to community, etc. were developed (cf. Greenhalgh ([9],[10]). In case of sexually transmitted diseases, such types of diseases are transmitted by heterosexual contacts. Various mathematical models constructed to understand the spread of *HIV/AIDS* amongst population of injecting drug users and treatment of *HIV/AIDS* (cf. Greenhalgh and Hay [11], Greenhalgh et al. [12], Donnelly and Cox [8], Srinivasa [22]). Other mathematical models related to control tuberculosis infection with *HIV/AIDS* were proposed (cf. Williams and Dye [25], Naresh and Tripathi [19], Wu [26]). Ball and Britton [2] developed a mathematical *SIR* model for a finite population and in this model two types of infections, exposure or mild infection, and severe infection were considered. They studied that how mild infection leads to severe infection. Clancy [6] formulated an *SIS* model for infectious disease transmission and an indirect transmission mechanism is incorporated. In this model a bi-variate Ornstein-Uhlenbeck approximation has been used to approximate the results. Wang and Zhao [23] developed an epidemic model with constant infection period and considered population dispersals between patches. It was found in this model that the longer infection period enhances the chances of spreading the disease. Chen [4] proposed an *SI* model to know the effect of voluntary vaccination on disease prevalence and it was found that the vaccine efficacy played an important role to control the disease. Whitman and Ashrafiuon [24] analyzed an infectious disease model using asymptotic theory and found that this method is more accurate in approximation and easy to understand for the model under consideration. Hiebeler [13] explored an *SIS* epidemiological model for population partitioned within and between the household populations. There is a recent literature that addresses the development of various mathematical models of infectious disease; to slow down the infection rate the different techniques viz global stability, explicit series solution, study of vaccination, bifurcation analysis etc. are used to solve the different mathematical models (cf. Buonomo et al. [3],

Mukandavire et al. [18], Khan et al. [14], Li and Wang [17], Qiu and Feng [20], Chow et al. [5].

In this paper, we have extended Greenhalgh model [10] by incorporating an extra class of mild infection and temporary immunization rate from mild infected class. We have assumed mild infection in place of exposed period. In *SIR* models, the population is divided into three compartments and transmissions between compartments are modeled. In this study, we include one more compartment for mild infected people. The population is divided into four classes namely individuals who are susceptible to the disease, but are not infected so far, mildly infected number of individuals, severely infected, who are able to spread the disease and those who have been removed by immunity (vaccination/ temporary immunity). In this study we assume population dependent death rate i. e. the death rate increases as population increases. The effect of vaccination on the disease is also considered.

The organization of rest of the paper is as follows. Section 2 describes the mathematical model by stating requisite assumptions and notations. The equilibrium analysis has been discussed in Section 3. In Section 4, the stability analysis has been provided. In Section 5, we deduce two special cases for (a) constant death rate and (b) fractional vaccinated population. Section 6 is given for the numerical analysis and discussion. Finally, the concluding remarks are given in Section 7.

2. The Mathematical Model. We consider a *SIR* model where the total population $N(t)$ at time t is divided into four classes as susceptible $S(t)$, mild infected $I_m(t)$, severe infected $I_s(t)$ and removed $R(t)$ so that at any time t, $N=S+I_m+I_s+R$. This model is similar to the model *SEIR* (susceptible, exposed, infected and removed), as mild infected is the exposed class which can catch the infection but can not transmit it. All newborns are assumed to be susceptible. We propose the following notation b is the birth rate; $f(N)$ the per capita population dependent mortality rate; β the per capita infection rate of an average susceptible individual, p the constant per capita vaccination rate of susceptible individuals; ν the rate at which individuals leave the mild infective class without immunity development and again become susceptible, γ the per capita rate at which individuals leave the mild infective class to removed class due to immunity against re-infection, ε the per capita rate at which individuals leave mild infective class to severities and α the per capita disease induced death rate. The graphical flow diagram of infectious disease is given in fig. 1.

The transition flow of population between various classes are described by the system of equations as follows

$$\frac{dS}{dt} = bN - \beta SI_m - f(N)S - pS + \nu I_m \ ,$$
$$\frac{dI_m}{dt} = \beta SI_m - f(N)I_m - \gamma I_m - \nu I_m - \varepsilon I_m \ ,$$
$$\frac{dI_s}{dt} = \varepsilon I_m - f(N)I_s - \alpha I_s \ ,$$
$$\frac{dR}{dt} = \gamma I_m - f(N)R + pS \ . \qquad (2.1)$$

On adding all the four equations, we get

$$\frac{dN}{dt} = bN - f(N)N - \alpha I_s \ .$$

3. Equilibrium Analysis. For equilibrium analysis, let us equate all the derivatives on the left hand side of set of equations (2.1) zero. We denote the equilibrium population of susceptible, mild infected, severe infected, recovered by immune and total population by S^*, I_m^*, I_s^*, R^* and N^* respectively. In view of this, we have

$$bN^* - \beta S^* I_m^* - f(N^*)S^* - pS^* + \nu I_m^* = 0,$$
$$\beta S^* I_m^* - f(N^*)I_m^* - \gamma I_m^* - \nu I_m^* - \varepsilon I_m^* = 0,$$
$$\varepsilon I_m^* - f(N^*)I_s^* - \alpha I_s^* = 0,$$
$$\gamma I_m^* - f(N^*)R^* + pS^* = 0 \ ,$$
$$bN^* - f(N^*)N^* - \alpha I_s^* = 0. \qquad (2.2)$$

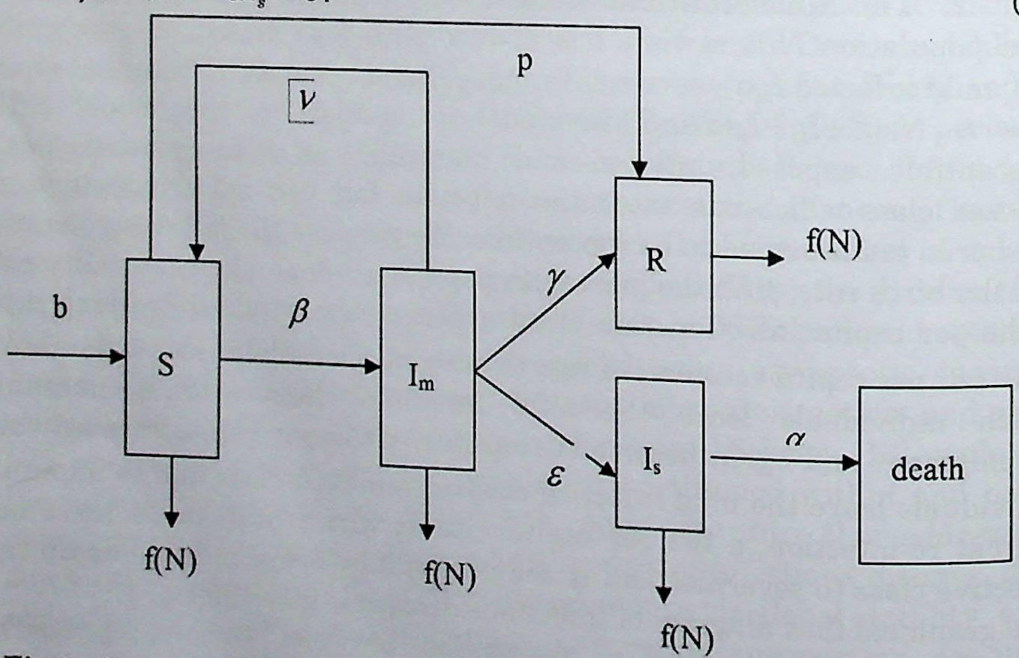

Fig.1: Graphical depiction of the transmission dynamics of the disease.

The limiting natural death rate is given by $f(\infty)=\lim_{N\to\infty} f(N)$.

Theorem 1. There are four equilibrium values as given below:

(i) Extinction of the population. $S^*=I_m^*=I_s^*=R^*=N^*=0$ (2.3)

(ii) When disease is not present.

$$S^*=\frac{b}{b+p}f^{-1}(b),\quad I_m^*=I_s^*=0,$$

$$R^*=\frac{p}{b+p}f^{-1}(b),\ N^*=f^{-1}(b) \tag{2.4}$$

(iii) If the population is free from severities.

$$S^*=\frac{b+\gamma+\nu+\varepsilon}{\beta},\ I_m^*=\frac{\beta b f^{-1}(b)-(b+\gamma+\nu+\varepsilon)(b+p)}{\beta(b+\gamma+\varepsilon)}, \tag{2.5a}$$

$$I_s^*=0,\qquad R^*=\frac{pS^*+\gamma I_{m1}}{b},\qquad N^*=f^{-1}(b) \tag{2.5}$$

(iv) The disease is present and maintains the population size at a level N^*.

$$S^*=\frac{f(N^*)+\gamma+\nu+\varepsilon}{\beta},\ I_m^*=\frac{(f(N^*)+\alpha)(b-f(N^*))}{\alpha\varepsilon}N^*,\ R^*=\frac{pS^*+\gamma I_m^*}{f(N^*)},$$

$$I_s^*=\frac{\varepsilon(b-f(N^*))(f(N^*)+\gamma+\nu+\varepsilon)(f(N^*)+p)}{\beta\left(\alpha\varepsilon b-(f(N^*)+\alpha)(b-f(N^*))(f(N^*)+\gamma+\varepsilon)\right)},$$

$$\frac{\beta N^*}{\alpha\varepsilon}=\frac{(f(N^*)+\gamma+\nu+\varepsilon)(f(N^*)+p)}{\alpha\varepsilon b-(f(N^*)+\alpha)(b-f(N^*))(f(N^*)+\gamma+\varepsilon)}\ .$$

The value of N^* must be positive and $b\ge f(N^*)$. .

If $\chi=f(N^*)$ then

$$\frac{\beta}{\alpha\varepsilon}f^{-1}(\chi)=\frac{(\chi+p)(\chi-e)}{(\chi-a)(\chi-c)(\chi-d)}, \tag{2.6}$$

where a,c and d be the roots of the cubic equation given below

$\chi^3+(\alpha+\gamma+\varepsilon-b)\chi^2+\left((\alpha+\gamma+\varepsilon)(\alpha-b)-\alpha^2\right)\chi-\alpha\gamma b$

and $e=-(\gamma+\nu+\varepsilon)$. We consider that $c,d>0>a$. Then it is evident that $a>e$.

Lemma 1. $g(\chi)\equiv\dfrac{(\chi+p)(\chi-e)}{(\chi-a)(\chi-c)(\chi-d)}$ is monotone decreasing in χ for $\chi\ge c$.

Proof. For proof see appendix.

Lemma 2. The susceptible and infected individuals will be positive only when $b\ge f(N^*)$.

Proof. It is clear from the equilibrium points that the susceptible individuals, infected individuals, mild infected individuals and severe infected individuals will be positive only if $b \geq f(N^*)$.

By virtue of Lemma 1, we note that

$\frac{\beta}{\alpha\varepsilon} f^{-1}(\chi) - \frac{(\chi+p)(\chi-e)}{(\chi-a)(\chi-c)(\chi-d)}$ is increasing expression in χ and zero at $\chi = f(N^*)$ using (15). Thus $b \geq f(N^*)$ if and only if

$$\frac{\beta}{\alpha\varepsilon} f^{-1}(b) \geq \frac{(\chi+\gamma+\nu+\varepsilon)(\chi+p)}{\alpha\varepsilon b - (\chi+\alpha)(b-\chi)(\chi+\gamma+\varepsilon)}$$

i. e. $f^{-1}(b) \geq \frac{(b+\gamma+\nu+\varepsilon)(b+p)}{\beta b}$.

With the help of equations (2.2) for equilibrium and above Lemma 1 and 2, we establish the condition, when the disease is present and maintains the population size at a level N^*

$$f^{-1}(b) > \frac{(b+\gamma+\nu+\varepsilon)(b+p)}{\beta b}.$$

4. Stability Analysis

4.1 The extinction of population. From the stability matrix the eigen values are obtained; which are: $-(f(0)+p)$, $-(f(0)+\gamma+\nu+\varepsilon)$, $-(f(0)+\alpha)$, $b-f(0)$. The first three eigen values are always negative whereas fourth one is negative only when $b \leq f(0)$. In this case the population is stable if $b \leq f(0)$, otherwise unstable.

4.2. For disease free population. In this case the eigen roots are $-(f(N^*)+p)$, $\beta S^* - (f(N^*)+\gamma+\nu+\varepsilon)$, $-(f(N^*)+\alpha)$, $-N^* f'(N^*)$.

All roots are always negative except second one. The second one root is negative if $\beta S^* \leq (f(N^*)+\gamma+\nu+\varepsilon)$. The population is stable if $\beta S^* \leq (f(N^*)+\gamma+\nu+\varepsilon)$ otherwise unstable. We conclude that the population is stable iff $f^{-1}(b) \leq \frac{(b+\gamma+\nu+\varepsilon)(p+b)}{\beta b}$, otherwise unstable.

4.3. The population is free of severe infection but mild infection exists. The characteristic equation is obtained as

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (4.1)$$

where λ is a characteristic root.

Here

$a_1 = \beta I_m^* + N^* f' + 2f + p + \alpha$,

$a_2 = N^* f'(f+\alpha) + (N^* f' + f + \alpha)(\beta I_m^* + f + p) + \beta(\beta S^* - \nu) I_m^*$,

$a_3 = N^* f'(f+\alpha)(\beta I_m^* + f + p) + +\beta(f+\alpha)(\beta S^* - \nu)N^* f T_m^* - \alpha\varepsilon f T_m^*$,

$a_4 = \beta(f+\alpha)(\beta S^* - \nu)N_1 f T_m^* - \alpha\varepsilon f T_m^*(\beta I_m^* + f + p) + \alpha\varepsilon\beta(b - S^* f')I_m^*$,

$f'(N^*) = f' = e$ and $f(N^*) = f$.

The coefficients of a biquadrate equation (16) give all roots with negative real part.

Routh-Hurwitch conditions are: $a_1>0$, $a_2>0$, $a_4>0$, $a_3\,(a_1\,a_2\text{-}a_3)>a_1^{\,2}\,a_4$
Now a_1, a_2, a_3, and a_4 can be written in the form as given below:

$a_1 = q_1 + eq_2,\ a_2 = u_1 + eu_2,\ a_3 = w_1 + ew_2,\ a_4 = x_1 + ex_2,$

where

$q_1 = \beta I_m^* + 2f + \alpha + p,\ q_2 = N^*$,

$u_1 = (\beta I_m^* + f + p)(f+\alpha) + \beta I_m^*(f+\gamma+\varepsilon),\ u_2 = q_1 N^*$,

$$w_1 = 0,\ w_2 = \left[(f+\alpha)\left((\beta I_m^* + f + p) + \beta I_m^*(f+\gamma+\varepsilon)\right) - \alpha\varepsilon\frac{I_m^*}{N}\right]N^*,$$

$$x_1 = \alpha\varepsilon\beta b I_m^*,\ x_2 = I_m^*\left[\beta(f+\alpha)(f+\gamma+\varepsilon) - \alpha\varepsilon\frac{(\beta I_m^* + \beta S + f + p)}{N}\right]N^*.$$

4.4 The infection exists whenever the population size is maintained at a level N^*. When disease exists and maintains the population size at a level N^*, we have $f^{-1}(b) > \dfrac{(b+\gamma+\nu+\varepsilon)(b+p)}{\beta b}$ and system is locally stable, otherwise unstable.

The characteristic equation obtained in this case is

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0, \tag{4.2}$$

where $b_1 = 3f + N^* f'(N^*) + p + \alpha + \beta I_m^* - b$,

$b_2 = (2f + N^* f' + \alpha - b)(f + p + \beta I_m^*) - (f+\alpha)(b - f - N^* f') - \alpha I_s^* f' + \beta I_m^*(f+\gamma+\varepsilon)$

$b_3 = (f+\alpha)(f + N^* f' - b)(\beta I_m^* + f + p) - \alpha I_s^* f'(\beta I_m^* + f + p) + \beta I_m^*(f+\gamma+\varepsilon)$
$(2f + N^* f' + \alpha - b)$,

$b_4 = \left\{\beta I_m^*(f+\gamma+\varepsilon)\left((f+\alpha) - \alpha I_s^* f'\right) - \alpha\varepsilon I_m^* f'(\beta I_m^* + f + p) - \alpha\varepsilon\beta(S^* f' - b)I_m^*\right\}$,

$f'(N^*) = f' = e$ and $f(N^*) = f$.

For the biquadrate equation (17), Routh-Hurwitch conditions are given by $b_1>0$, $b_2>0$, $b_4>0$, $b_3\,(b_1 b_2 - b_3) > b_1^{\,2}\, b_4$.

4.5 Remarks

(i) If equilibrium conditions are satisfied for disease free population in case when the population maintains itself at a level N_1, then the number of

susceptible individuals is given by $\frac{b}{b+p} f^{-1}(b)$. Now we look into the effect of one infected individual who is introduced into the population. This infected introduced person may die at the rate $f(N_1)$, however after permanent immunization he goes to removed class at the rate γ, due to temporary immunization he leaves the infected class at the rate ν and goes to severe infected class with rate ε. Infected person leaves the mild infected class at the rate $f(N_1)+\gamma+\nu+\varepsilon$ and average infection period is $T = \frac{1}{f(N^*)+\gamma+\nu+\varepsilon}$. The expected number of secondary infections produced by one primary infection is $\beta S^* T = \beta b f^{-1}(b)/(b+p)$ $(f(N^*)+\gamma+\nu+\varepsilon)$. This is equal to the reproduction number R_0. If this quantity is greater than one then the disease free equilibrium is unstable. It depends on the specific disease and on the rate of contacts, which may depend on the population density. If $R_0<1$, the disease will die out, but if this quantity is greater than 1 the disease will be endemic.

(ii) In order to prevent a disease from becoming endemic, it is essential to reduce the basic reproductive number below 1. This may be achieved by vaccination process. If newborn members per unit time of the population are successfully immunized then the rate of immunization should be at least $b(R_0\text{-}1)$. If a fraction $p/p+b$ of the bN_1 newborn members per unit time of the population is successfully immunized, the effect is to replace N_1 by $N_1\{1-(p/(p+b))\}$ and reduces the basic reproductive number to $R_0\{1-(p/(p+b))\}$. The proportion of the population which is immunized is $p/(p+b)=1-(1/R_0)$. If a large enough fraction has been immunized then the disease can not become endemic.

(iii) We can draw some other inferences from equilibrium values. The condition $\beta S^* < f(N^*)+\gamma+\nu+\varepsilon$ is for disease-free population and population is locally stable about small perturbation if $\beta S^* > f(N^*)+\gamma+\nu+\varepsilon$, otherwise the population is unstable. If the disease persists then ratio $\frac{\beta S^*}{f(N^*)+\gamma+\nu+\varepsilon} > 1$, i. e. if $\beta S^* > f(N^*)+\gamma+\nu+\varepsilon$ then the mild infection leads to severe infection and disease becomes endemic; if $\frac{\beta S^*}{f(N^*)+\gamma+\nu+\varepsilon} < 1$ then the mild infection dies out and it does not lead to severe infection.

(iv) Now we predict effect of other parameters viz ν, p, γ and ε on equilibrium values in case of disease prevalence. On increasing vaccination rate (p), the susceptible population remains constant but total number of population and severe infected population increases. If ν increases then equilibrium susceptible individuals (S_1), the total equilibrium population (N^*), mild infected population and severe infected population increase; the removed population increases sum of p times in susceptible population and γ times in mild infected population. Thus it is clear that the rate of increment in removed population is higher. It is straightforward that on increasing ε, the severe infected population increases but mild infected population decreases. This quantity enhances susceptible population also. It is also noted that disease induced death increases as ε increases. If we consider $b=f(N^*)$, then both types of infections tend to die away.

(v) The Case when $b>f(\infty)$. In this case, in the absence of disease, the birth rate exceeds the death rate. The population size increases without limit and becomes infinite. So if single infected case is introduced into the susceptible class and the infection spreads into the susceptible class, the disease will spread. Equation (14) which determines the population size has a unique root if and only if $f(\infty)>c$.

Now in the presence of disease when the vaccination rate is zero (i.e. $p=0$), following three cases arise:

(a) $b \le f(0)$. this case represents only one equilibrium condition when the population dies out. This equilibrium condition is locally stable to small perturbation.

(b) $b>f(0)$ and $f(\infty)>c$. In this case two equilibria arise. First one is the population extinction. This equilibrium is unstable to small perturbation. Second one is the case when disease maintains population size at a level N_1 and this is determined by equation (2.5A). The equilibrium is stable to small perturbation.

(c) $f(\infty)\le c$ and $b>f(0)$. For this case, there is only one possibility of equilibrium where the population dies out and this is locally unstable to small perturbation. In this case, if there is one person, then the population increases infinitely; if only one infected individual is introduced in the susceptible class, then in the population we find the fractions of (i) mild infected as $\dfrac{(f(\infty)+\alpha)(b-f(\infty))}{\alpha\varepsilon}$ and (ii) severe infected as $\dfrac{((b-f(\infty))}{\alpha}$; also the proportion of immune population is 1-$\left[(f(\infty)+\alpha)(b-f(\infty))/\alpha\varepsilon\right]$ and the total number of susceptible individuals is $\dfrac{f(\infty)+\gamma+\nu+\varepsilon}{\beta}$.

5. Special Cases. Now we shall discuss some cases as particular cases of our model as follows: Case when the natural death rate is constant. The equations governing the model for constant death rate (μ) are

$$\frac{dS}{dt} = bN - \beta S I_m - \mu S - pS + \nu I_m \ ,$$

$$\frac{dI_m}{dt} = \beta S I_m - \mu I_m - \gamma I_m - \nu I_m - \varepsilon I_m \ ,$$

$$\frac{dI_s}{dt} = \varepsilon I_m - \mu I_s - \alpha I_s \ ,$$

$$\frac{dR}{dt} = \gamma I_m - \mu R + pS \ ,$$

$$\frac{dN}{dt} = bN - \mu N - \alpha I_s . \qquad (5.1)$$

Equilibrium analysis. The equilibrium values are obtained by solving the following equations

$$bN^* - \beta S^* I_m^* - \mu S^* - pS^* + \nu I_m^* = 0,$$

$$\beta S^* I_m^* - \mu I_m^* - \gamma I_m^* - \nu I_m^* - \varepsilon I_m^* = 0,$$

$$\varepsilon I_m^* - \mu I_s^* - \alpha I_s^* = 0,$$

$$\gamma I_m^* - \mu R^* + pS^* = 0,$$

$$bN^* - \mu N^* - \alpha I_s^* = 0. \qquad (5.2)$$

Theorem 2. For equilibrium points S^*, I_m^*, I_s^*, R^* and N^*, the following five possibilities arise.

(i) The population dies out if $S^* = I_m^* = I_s^* = R^* = N^* = 0$. This is always possible. If $b < \mu$, then the equilibrium is stable to small perturbation.

(ii) When $b = \mu$.

$S^* = N^*$ and $I_m^* = I_s^* = R^* = 0$, the equilibrium is locally stable to small perturbation if $\beta N^* \le \mu + \gamma + \nu + \varepsilon$, otherwise unstable.

(iii) When $b = \mu$.

$N^* = S^* + I_m^*$ and $I_s^* = R^* = 0$

Same conditions occur as in case (ii).

(iv) When $b \ge \mu$ and $\mu^2\left(\mu + 2\alpha + \varepsilon + \dfrac{\alpha\varepsilon}{\mu}\right) > b(\mu^2 + \alpha^2 + \mu\varepsilon)$.

The disease is present but population size should be at a level of N_1. On solving the equations from (18)-(22), the equilibrium points are

$$S^* = \frac{\mu + \gamma + \nu + \varepsilon}{\beta},\ I_m^* = \frac{(\mu + \alpha)(b - \mu)}{\alpha\varepsilon} N^*,\ R_1 = \frac{pS^* + \gamma I_m^*}{\mu},$$

$$I_s^{\circ} = \frac{\varepsilon(b-\mu)(\mu+\gamma+\nu+\varepsilon)(\mu+p)}{\beta\left(\alpha\varepsilon b-(\mu+\alpha)(b-\mu)(\mu+\gamma+\varepsilon)\right)},$$

The size of population is given by

$$\frac{\beta N^{*}}{\alpha\varepsilon} = \frac{(\mu+\gamma+\nu+\varepsilon)(\mu+p)}{\alpha\varepsilon b-(\mu+\alpha)(b-\mu)(\mu+\gamma+\varepsilon)}. \tag{5.3}$$

The equilibrium is stable to small perturbation.

(v) When $b \geq \dfrac{\mu^2\left(\mu+2\alpha+\varepsilon+\dfrac{\alpha\varepsilon}{\mu}\right)}{(\mu^2+\alpha^2+\mu\varepsilon)}$.

The population grows without limit. There is a situation when the infected individuals will become constant fraction of total population and susceptible individuals will also become constant i. e. $(\mu+\gamma+\nu+\varepsilon)/\beta$.

Interpretation. Here there are four cases

(i) If $b<\mu$, then the population will die out.

(ii) If $b=\mu$, in this case the population will be at constant level N^*. The threshold population size is $N^* = \dfrac{\mu+\gamma+\nu+\varepsilon}{\beta}$. It is clear that in the absence of disease the threshold condition is independent of vaccination rate.

(iii) If $b>\mu$, then in the absence of disease the population grows without limit. In this case the population size will become infinite and reproduction ratio will be infinite. Now to maintain the population size equilibrium, we introduce single infected case in the population. Then the disease will establish itself. A model was developed by Dietz ([7] in which the death rate was considered constant but vaccination campaign was not included. In this investigation, we consider the effect of vaccination campaign on the population.

(iv) If $b \geq \dfrac{\mu^2\left(\mu+2\alpha+\varepsilon+\dfrac{\alpha\varepsilon}{\mu}\right)}{(\mu^2+\alpha^2+\mu\varepsilon)}$, the population grow infinitely and the population growth is independent of the vaccination rate.

From the equilibrium values, we deduce some inferences as follows: on increasing p, the total equilibrium population increases, the number of both type of infected individuals increases, but susceptible individuals remain constant and it is clear that the increment is in immune class.

Case II. Fractional vaccinated population. The model is governed by following equations:

$$\frac{dS_a}{dt} = bNq - \beta S_a I_m - f(N)S_a - pS_a + \nu I_m ,$$
$$\frac{dS_b}{dt} = bN(1-q) - \beta S_b I_m - f(N)S_b ,$$
$$\frac{dI_m}{dt} = \beta S I_m - f(N)I_m - \gamma I_m - \nu I_m - \varepsilon I_m ,$$
$$\frac{dI_s}{dt} = \varepsilon I_m - f(N)I_s - \alpha I_s ,$$
$$\frac{dR}{dt} = \gamma I_m - f(N)R + pS_a ,$$
$$\frac{dN}{dt} = bN - f(N)N - \alpha I_s \ . \tag{5.3}$$

Equilibrium analysis. To obtain the equilibrium values, we solve the following equations:

$$bN^* q - \beta S_a^* I_m^* - f(N^*)S_a^* - pS_a^* + \nu I_m^* = 0 ,$$
$$bN^*(1-q) - \beta S_b^* I_m^* - f(N^*)S_b^* = 0 ,$$
$$\beta S^* I_m^* - f(N^*)I_m^* - \gamma I_m^* - \nu I_m^* - \varepsilon I_m^* = 0 ,$$
$$\varepsilon I_m^* - f(N^*)I_s^* - \alpha I_s^* = 0 ,$$
$$\gamma I_m^* - f(N^*)R^* + pS_a^* = 0 ,$$
$$bN^* - f(N^*)N^* - \alpha I_s^* = 0 \ . \tag{5.4}$$

Theorem 3. There are following three possibilities for equilibrium analysis.

(i) The population has died out.

$S_a^* = S_b^* = I_m^* = I_s^* = R^* = N^* = 0$ always possible. (5.5)

(ii) When disease is not present.

The equilibrium values are

$$S_a^* = \frac{bq}{f(N^*)+p} N^*, \quad S_b^* = (1-q)N^*,$$
$$R^* = \frac{pq}{f(N^*)+p} N^*, \quad N^* = f^{-1}(b) \ . \tag{5.6}$$

(iii) Disease exists and regulates the size of population at a level N^*.

In this case $b \geq f(N^*)$, the equilibrium values are

$$I_s^* = \frac{b - f(N^*)}{\alpha} N^*, \ I_m^* = \frac{(f(N^*)+\alpha)(b - f(N^*))}{\alpha\varepsilon} N^*,$$
$$S_b^* = \frac{b(1-q)}{(\beta/\alpha\varepsilon)\left((f(N^*)+\alpha)(b-f(N^*))N^*\right) + f(N^*)} N^* \quad ,$$

$$S_a^* = \frac{f(N^*)+\gamma+\nu+\varepsilon}{\beta} - \frac{b(1-q)N^*}{(\beta/\alpha\varepsilon)(f(N^*)+\alpha)(b-f(N^*))N^* + f(N^*)}.$$

The equation in N^* is given as follows:

$$\left[\left(b+\frac{\nu}{\alpha\varepsilon}(f+\alpha)(b-f)\right)N^* - \frac{f+\gamma+\nu+\varepsilon}{\beta}\left(\frac{\beta(f+\alpha)(b-f)N^*}{\alpha\varepsilon}+f+p\right)\right]$$

$$\times\left[\frac{\beta(f+\alpha)(b-f)N^*}{\alpha\varepsilon}+f\right] = -bp(1-q)N^*\ , \qquad (5.7)$$

where $f(N^*) = f$

Lemma 3. $\alpha\varepsilon b > (\gamma+\varepsilon+\chi)(\varphi+\alpha)(b-\chi)$.

Proof. See appendix

Lemma 4. Let c be the unique positive root of $\alpha\varepsilon b-(\gamma+\varepsilon+\chi)(\chi+\alpha)(b-\chi)=0$. Then for each χ with $c<\chi\le b$, there is unique positive $N^*(\chi)$ satisfying (5.7) and $N^*(\chi)$ tends to infinity as χ tends to c.

Proof. See appendix.

Lemma 5. $N(\chi)$ is strictly monotone decreasing in χ for $c<\chi\le b$.

Proof. For proof see Greenhalgh [10].

From the Lemma 2 and equation (5.7), we see that model has a unique positive root for N^* if $b\le f(N^*)$ if and only if $f(\infty)>c$ and either $b>f(\infty)$ or $f^{-1}(b)<N^*(b)$ otherwise it has no positive roots. If we put $f=b$ in equation (5.7), we get

$$N^*(b) = \frac{(b+\gamma+\nu+\varepsilon)(b+p)}{\beta\big(b+p(1-q)\big)}\ . \qquad (5.8)$$

The threshold value is

$$R_0 = \frac{\beta f^{-1}(b)\big(b+p(1-q)\big)}{(b+\gamma+\nu+\varepsilon)(b+p)}\ .$$

The threshold value in the absence of vaccination (i. e. $p=0$) is

$$R_1 = \frac{\beta f^{-1}(b)}{(b+\gamma+\nu+\varepsilon)}\ .$$

Stability analysis.

(i) The population dies out. The eigen values are $-(f(0)+p), -f(0), -(f(0)+\gamma+\varepsilon), -(f(0)+\alpha), b-f(0)$. All eigen values are negative except fifth one. Thus the equilibrium is locally stable if $b\le f(0)$ otherwise unstable.

(ii) Disease free equilibrium. In this case the Eigen values are:

$$-(f(N^*)+p),\ -f(N^*),\ \big(\beta S^* - (f(N^*)+\nu+\gamma+\varepsilon)\big),\ -(f(N^*)+\alpha),\ -N^*f(N^*).$$

All eigen values are always negative except λ_3. Thus the equilibrium is locally stable if $\beta S^* \le (f(N^*)+\gamma+\nu+\varepsilon)$, otherwise unstable. With the help of the equilibrium values, we get that the equilibrium is locally stable if only if

$$f^{-1}(b) \le \frac{(b+\gamma+\nu+\varepsilon)(b+p)}{\beta[b+p(1-q)]}.$$

(iii) When disease is there and maintains the population size at a level N^*. This is possible if and only if

$$f^{-1}(b) > \frac{(b+\gamma+\nu+\varepsilon)(b+p)}{\beta[b+p(1-q)]}.$$

The equilibrium is locally stable if above condition exists, otherwise locally unstable.

Interpretation. Here there are two threshold values, namely R_0 which represents the actual threshold value and R_1 which is the threshold value in absence of the vaccination. The threshold value is given by $R_0 = \frac{\beta f^{-1}(b)(b+p(1-q))}{(b+\gamma+\nu+\varepsilon)(b+p)}$, where the population is in steady state and disease dies out. Therefore to maintain the population in equilibrium, the single infected person's entry in the population is taken into account. This infected person produces secondary cases of infections i. e. disease will take off. We deduce that the persistence or non-persistence of disease depends on the reproduction number.

The relationship between R_0 and R_1 is given as $R_0 = R_1\left(1-pq/(p+b)\right)$. If $R_1>1$, then in this case (without vaccination) the disease may persist. The question is to eradicate the disease, for this to meet population must be vaccinated. But the answer of the question depends on the proportion of q. If this proportion is $q<(1\text{-}1/R_1)$, then removal of the disease is impossible i. e. disease exists in the population. If $q>1\text{-}(1/R_1)$, then the vaccination rate must be $(b(R_1\text{-}1)/\ (1\text{-}R_1(1\text{-}q)))$ to reach the desired goal.

6. Numerical Analysis and Discussion. The numerical simulation and stability analysis of equations (1) to (5) have been given in this section. The equations are numerically solved with the help of Range-Kutta fourth order method by fixing different parameters in MATLAB software as follows. The default parameters are : $b=2.5$, $\beta_1=3.86$, $p=0.2$, $\upsilon=0.1$, $\varepsilon=0.51, \gamma=0.5$, $\alpha=1.0$, $\mu_1=0.01$, $\mu_2=0.0001$. The function $f(N)$ is linear function in N; therefore let $f(N)=\mu_1+\mu_2 N$. β is per capita infection rate

so that $\beta = \beta_1 / N$, where β_1 is the infection rate. The initial values are $N(0)=21500$, $S(0)= 18320$, $Im(0)=1000$, $Is(0)=180$, and $R(0)=2000$.

Fig. 2 depicts distribution of population with time for different classes when b=0. It is noticed that all the classes of population converge to zero as time increases i. e. all the population dies out after some time. It is also seen that susceptible and removal populations decrease continuously but infected mild and severe populations increase a little for a short while and then tends to be constant because $b=0$ and some mild infection develops to severity and due to severity, severe infected population dies out by disease-induced death. Fig.3 shows the variation of population in different classes with time; in this figure we see that population of all classes increases for a short period after that population of all classes becomes constant (stable).

In case of $b \neq 0$ figs. 4-7 show the variation of susceptible, mild infected, severe infected and removal populations respectively, with time for different infection rates. In fig.4 we see that on increasing infection rate the susceptible population initially increases but shortly it decreases and then becomes stable after some time. Fig. 5 shows that mild infection increases as infection rate increases. Fig.6 depicts that severe infection increases with infection rate and infection becomes endemic and disease always persists in the population. The removable population also increases as infection rate increases (fig.7).

Figs. 8-10 are given for variation of mild infected, severity class and removal class with time for different values of ε. In fig 8, the mild infection decreases on increasing ε. It is noticed from fig. 9 that the increment in ε the population of severity class initially increases and after some time this population decrease sharply because disease-induced death rate. Removal class decreases with time as ε increases (fig. 8).

Figs. 11-14 depict the variation of mild infected population with time for different vaccination rates. In fig. 11, the susceptible population initially decreases sharply on increasing vaccination rate p but after some time it becomes stable. From figs.12 and 13, we see that both the infectious populations, mild and severe, decrease as increasing vaccination rate and after some time infection dies out. From fig.14, it is noticed that the removable population increases sharply with time on increasing vaccination rate.

Overall, we observe by these figures that the infection can be controlled or even eradicated by vaccination.

7. Conclusion. In this model, we investigate the role of vaccination for the population in which disease is taking off and there are three

equilibrium points viz. population extinction, disease free-population and disease exists whenever maintain the population at the steady level. All these three equilibrium cases are stable for small perturbations. In case when disease in the population tends to endemic then the vaccination campaign is used to eradicate the disease. The vaccination rate for eradication the disease must be $b(R_0\text{-}1)$, when R_0, the threshold value exceeds one. According to the equilibrium removed value, the fraction of the population that is vaccinated is $p/(p+b)=1\text{-}1/R_0$. The temporary immunization also enhances the removed class of the population.

In the case of constant death rate, the susceptible population is independent of the vaccination rate but total population, both types of infections and removed class population increase. If $b>f(\infty)$ the population increases infinitely.

We know that the total population can not be vaccinated and the vaccination can not be cent-percent effective so real situation is that the vaccination is effective on a proportion of the total population. In this case two reproduction numbers arise of which one is for vaccinated population and another one is for non-vaccinated population. Thus existence of disease depends on these two numbers.

Based on numerical simulation, it is noticed that vaccination enhances the removed population and that leads to eradication of the infection. The rate of transfer from mild to severity (ε) also affects the infectious population significantly. Numerical simulation validates the analytical results satisfactorily.

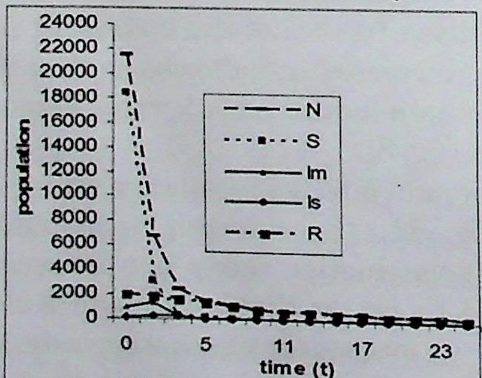

Fig. 2: Variation of population in different classes with time when $b=0$.

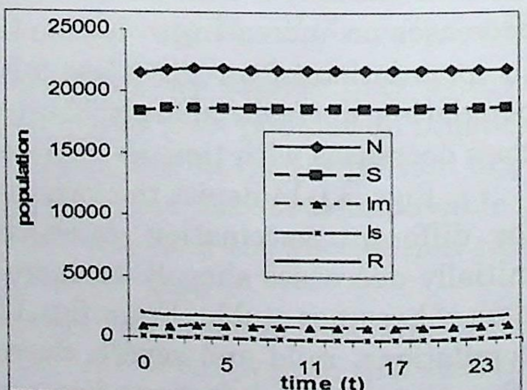

Fig. 3: Variation of population in different classes with time.

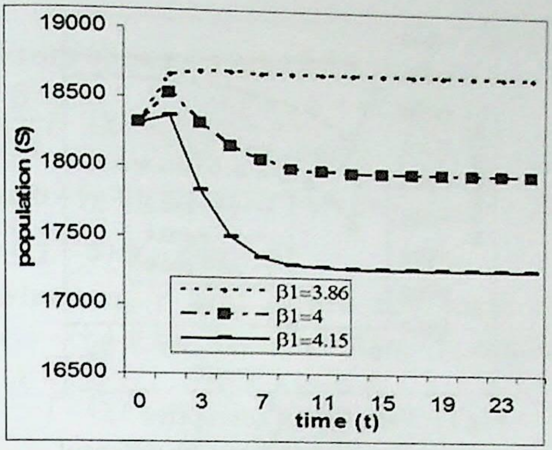

Fig. 4: Variation of susceptible population (S) with time for different infection rates(β).

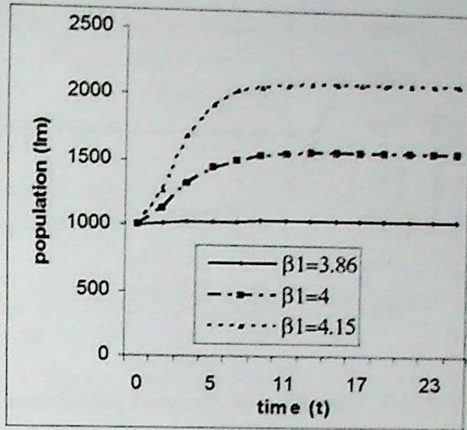

Fig.5: Variation of mild infective population (I_m) with time for different infection rates (β)

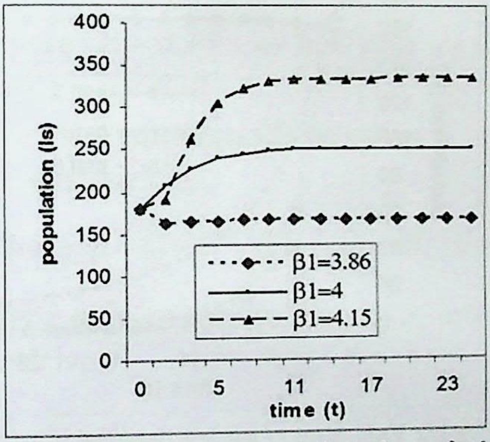

Fig. 6: Variation of severe infected population (Is) with time for different infection rates (β).

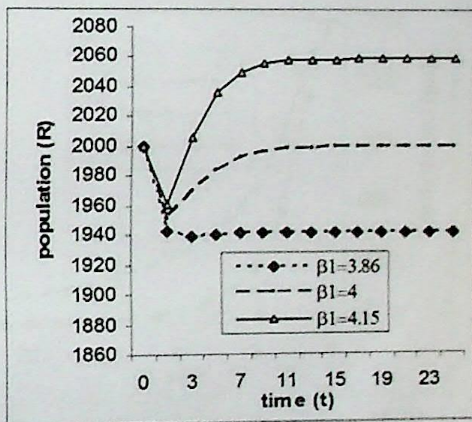

Fig. 7: Variation of removal population (R) with time for different infection rates (β)

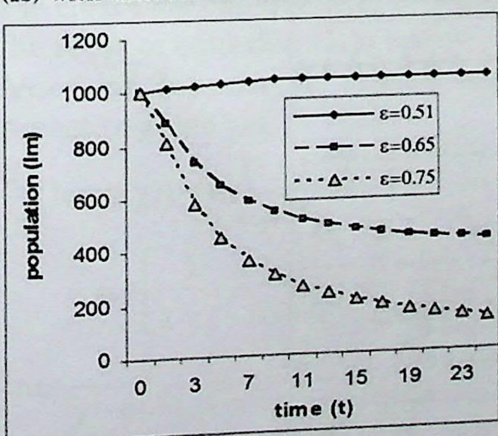

Fig. 8: Variation of mild infected population (Im) with time for different values of ε.

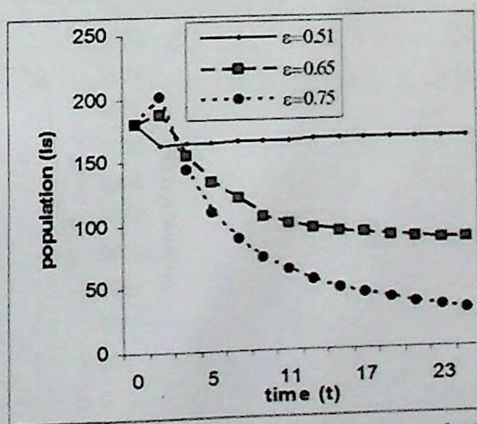

Fig. 9: Variation of severe infected population (Is) with time for different values of ε.

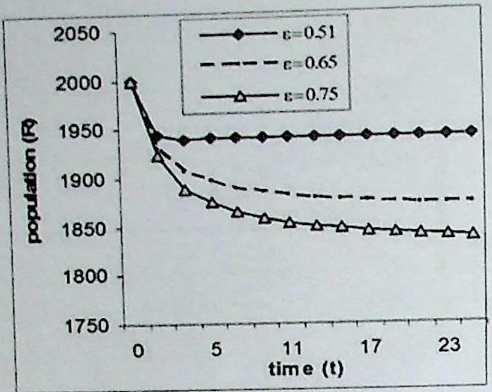

Fig. 10: Variation of removal population (R) with time for different values of ε.

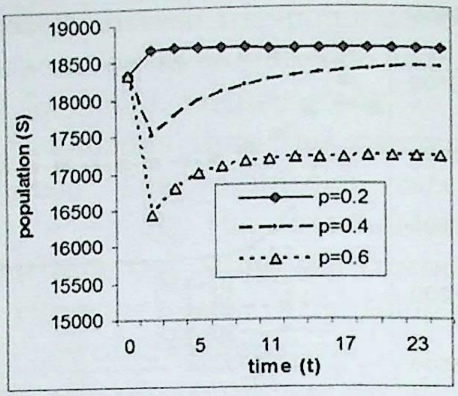

Fig.11: Variation of susceptive population (S) with time for different vaccination rates p.

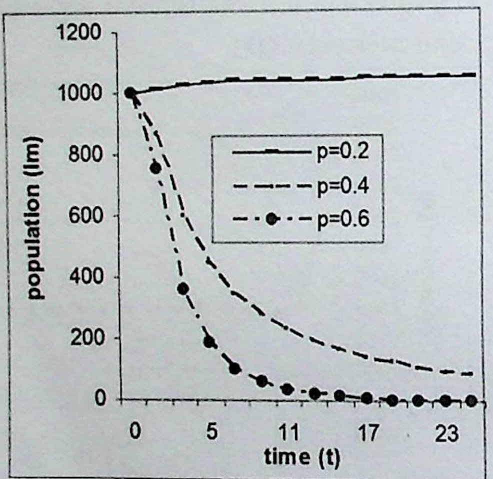

Fig. 12: Variation of mild infective population (I_m) with time for different vaccination rates p

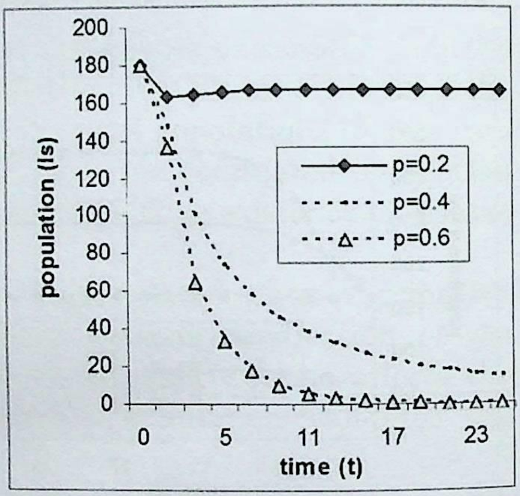

Fig.13: Variation of severe infected population (Is) with time for different vaccination rates p

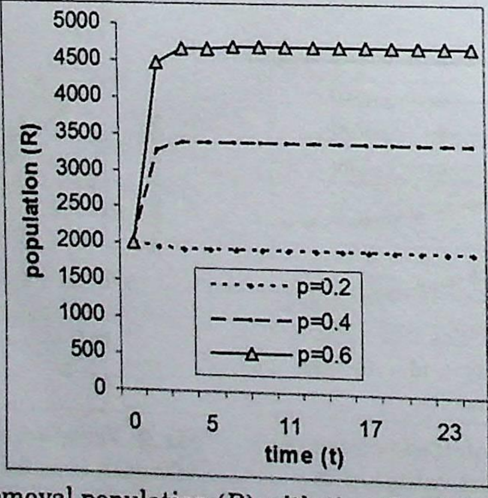

Fig. 14: Variation of removal population (R) with time for different vaccination rates p

Appendix

From the equation (2.6), we have

$$\frac{\beta}{\alpha\varepsilon} f^{-1}(\chi) = \frac{(\chi+\gamma+\nu+\varepsilon)(\chi+p)}{\alpha\varepsilon b-(\chi+\alpha)(b-\chi)(\chi+\gamma+\varepsilon)}, \tag{1}$$

$$\alpha\varepsilon b-(\chi+\alpha)(b-\chi)(\chi+\gamma+\varepsilon) = \chi^3+(\alpha+\gamma+\varepsilon-b)\chi^2+\left[(\alpha+\gamma+\varepsilon)(\alpha-b)-\alpha^2\right]\chi-\alpha\gamma b$$
$$\equiv (\chi-a)(\chi-c)(\chi-d),$$

where a, c and d are the roots of cubic expression and we assume $e=-(\gamma+\nu+\varepsilon)$ then equation (1) can be written as

$$\frac{\beta}{\alpha\varepsilon} f^{-1}(\chi) = \frac{(\chi+p)(\chi-e)}{(\chi-a)(\chi-c)(\chi-d)} \quad .$$

We consider that $c,d>0>a$. Then it is very clear that $a>e$.

The solution of following cubic equation as

$$\chi^3+B\chi^2+C\chi+D=0 \tag{2}$$

where $B=\alpha+\gamma+\varepsilon-b,\ C=(\alpha+\gamma+\varepsilon)(\alpha-b)-\alpha^2,\ D=-\alpha\gamma b$.

Let $\chi = u - B/3$

then the equation (2) becomes

$$u^3+p^4+q=0,$$

where $p=C-B^2/3, q=D-BC/3+2B^3/27=-\frac{1}{27}\left[27\alpha\gamma b+\beta(2p+3c)\right]$,

Here p is negative, also q is negative, so $p^3/27$ is negative and $q^2/4$ is positive.

$$y^3=-q/2+\left(q^2/4+p^3/27\right)^{1/2} \text{ and } z^3=-\frac{q}{2}-\left(\frac{q^2}{4}+\frac{p^3}{27}\right)^{1/2}.$$

It is clear that the quantity B is positive, C is negative and D is negative.

The roots of equation (1.2) are $c=y+z-B/3$, $d=wy+w^2z-B/3$, $a=w^2y+wz-B/3$.

Proof of lemma 1. Taking logarithms and then differentiating with respect to χ, we get

$$\frac{d}{d\chi}(\log g(\chi)) = \frac{1}{\chi+p}+\frac{1}{\chi-e}-\frac{1}{\chi-a}-\frac{1}{\chi-c}-\frac{1}{\chi-d}$$
$$= -\left[\frac{1}{\chi-c}-\frac{1}{\chi+p}\right]-\left[\frac{1}{\chi-a}-\frac{1}{\chi-e}\right]-\frac{1}{\chi-d}$$

Since $\frac{1}{\chi-c}>\frac{1}{\chi+p}, \frac{1}{\chi-a}>\frac{1}{\chi-e}$.

So that $\frac{d}{d\chi}(\log g(\chi))<0$.

Thus $g(\chi)$ is monotone decreasing in χ.

It is very clear that $g(\chi)$ is negative for $0<\chi<c$ and positive for $\chi>c$. $\frac{\beta}{\alpha\varepsilon}f^{-1}(\chi)$ is defined for $f(0)\le\chi<f(\infty)$ is always positive and monotone increasing where it is defined, 0 at $f(0)$ and tends to infinity as χ tends $f(\infty)$. Hence if $f(\infty)>c$ equation (1) has a unique positive solution for χ whereas if $f(\infty)<c$ the equation (1) has no positive solution for χ.

Proof of Lemma 3.

$$bN^* > \beta S^* I_m^* = \frac{(\chi+\alpha)(b-\chi)N^*}{\alpha\varepsilon}$$

Let χ be fixed as $f(N^*)=\chi$ and $\chi\le b$. Then

$$F(N^*,\chi)\equiv\left[\left(b+\frac{\nu}{\alpha\varepsilon}(f+\alpha)(b-f)\right)N^*-\frac{\chi+\gamma+\nu+\varepsilon}{\beta}\left(\frac{\beta(\chi+\alpha)(b-\chi)N^*}{\alpha\varepsilon}+\chi+p\right)\right]$$

$$\times\left[\frac{\beta(\chi+\alpha)(b-\chi)N^*}{\alpha\varepsilon}+\chi\right]=0$$

with roots for N^* as given by

$$\omega_1=\frac{\alpha\varepsilon(\gamma+\varepsilon+\nu+\chi)(\chi+p)}{\beta\left[\alpha\varepsilon b-(\gamma+\varepsilon+\chi)(\chi+\alpha)(b-\chi)\right]} \quad\text{and}\quad \omega_2=-\frac{\alpha\varepsilon\chi}{\beta(\chi+\alpha)(b-\chi)}$$

Now ω_1 is positive and ω_2 is negative. Thus $N^*(\chi)$ has unique positive root.

Proof of Lemma 4. It is clear that $F(N^*,\chi)+bp(1-q)N^*$ is quadratic in N^* as seen from following:

$$\frac{\beta}{\alpha\varepsilon}(\chi+\alpha)(b-\chi)\left[b-\frac{(\chi+\alpha)(b-\chi)(\chi+\gamma+\varepsilon)}{\alpha\varepsilon}\right]N^{*2}$$

$$+\left[b+\frac{(\chi+\alpha)(b-\chi)}{\alpha\varepsilon}\big(\nu\chi-(\chi+\nu+\gamma+\varepsilon)(2\chi+p)\big)\right]N^*-\chi(\chi+p)\frac{(\chi+\nu+\gamma+\varepsilon)}{\beta}=0$$

$$X(\chi)N^{*2}+Y(\chi)N^*+Z(\chi)=0$$

where $X(\chi)=\frac{\beta}{\alpha\varepsilon}(\chi+\alpha)(b-\chi)\left[b-\frac{(\chi+\alpha)(b-\chi)(\chi+\gamma+\varepsilon)}{\alpha\varepsilon}\right]$

Proof of Lemma 5. (As proof given by Greenhalgh [10].

Let $c<\chi_1<\chi_2\le b$. For monotone decreasing of $N(\chi)$ in χ, we have to prove that $N(\chi_1)>N(\chi_2)$.

From the Lemma 1, ω_1 is monotone decreasing in χ and ω_2 is also monotone decreasing in χ. So $\omega_1(\chi_1)>\omega_1(\chi_2)$ and $\omega_2(\chi_1)>\omega_2(\chi_2)$. $F(N,\chi)$ does not depend on q. Let $N(\chi_1)$ is the unique positive root of

$F(N, \chi_1) = -bp(1-q)N$ and $N(\chi_2)$ be the unique positive root of $F(N, \chi_2) = -bp(1-q)N$. $F(N, \chi_1)$ and $F(N, \chi_2)$ are quadratic in N and cut at most twice. $F(N, \chi_2) > F(N, \chi_1)$ at ω_1 and $F(N, \chi_2) < F(N, \chi_1)$ at ω_2. In the region $\omega_2(\chi_2) \leq N \leq \omega_1(\chi_1)$ at $N=Z$ (say). $F(N, \chi_2) > F(N, \chi_1)$ $\Leftrightarrow N > Z$. It is clear that $N_1 < N_2 \Leftrightarrow N_1 > Z$. For no-vaccination ($q=0$) case N_1 is smallest.

For no vaccination case

$$N(\chi) = \frac{\alpha\varepsilon\chi(\chi+\gamma+\nu+\varepsilon)}{\beta(\alpha\varepsilon b-(f(N^*)+\alpha)(b-f(N^*))(f(N^*)+\gamma+\varepsilon))}$$

Expression (3) is monotone decreasing in χ by lemma 1. Hence for $q=0$, $N_1 > N_2 \Rightarrow N_1, N_2 > Z$. So for all q $N_1, N_2 > Z$ and $N_1 > N_2$.

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Jñānābha , Vol. 45, 2015
(Dedicated to Honor Professor H.M. Srivastava on his Platinum Jubilee Celebrations)

APPLICATION OF $\bar{H}$-FUNCTION IN ELECTRIC CIRCUIT THEORY

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ABSTRACT

On the basis of the usefulness and great importance of the differential equations in Electric circuit theory problems, the authors find out the solution of a differential equation with an objective to obtain the value of the charge at any time t in a simple electric circuit consisting of resistance, inductance, capacitance and a source of electromotive force $E_0P(t)$, when $P(t)$ is taken in terms of the Inayat Hussain $\bar{H}$-function [4,5]. This function is quite general in nature because it includes a number of well known elementary functions as its special cases. Several known and new results can be established by the main result, occurring frequently in mathematical physics and engineering.

2010 Mathematics Subject Calssification : 33C6.

Keywords: *H*-function, Electric circuit theory, Resistance, Inductance, Capitance, Electricmotive force.

1. Introduction. The $\bar{H}$-function [1] occurring in the paper is defined as

$$\bar{H}_{P,Q}^{M,N}[z] = \bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,N}, & (a_j,A_j)_{N+1,P}\\ (b_j,B_j)_{1,M} & , (b_j,B_j;\beta_j)_{M+1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\bar{\phi}(s)z^s ds \quad \ldots(1.1)$$

where

$$\bar{\phi}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j - B_j s)\prod_{j=1}^{N}\{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+B_j s)\}^{\beta_j}\prod_{j=N+1}^{P}\Gamma(a_j - A_j s)}, \quad \ldots(1.2)$$

which contains the fractional powers of some of the gamma functions. Here and throughout the paper $a_j(j=1,\ldots,P)$ and $b_j(j=1,\ldots,Q)$ are complex parameters, $A_j \geq 0\ (j=1,\ldots,P)$, $B_j \geq 0\ (j=1,\ldots,Q)$ (not all zero simultaneously) and the exponents $\alpha_j(j=1,\ldots,N)$ and $\beta_j(j=M+1,\ldots,Q)$ can take non-integer values.

The contour in (1.1) is imaginary axis $Re(s)=0$. It is suitably indented in order to avoid the singularities of the Gamma functions and to keep those singularities on appropriate sides. Again, for $\alpha_j(j=1,\ldots,N)$ not an integer, the poles of the gamma functions of the numerator in (1.2) are converted to the branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - B_j s)\ (j=1,\ldots,M)$ and $\Gamma(1-a_j+A_j s)\ (j=1,\ldots,N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

Buschman and Srivastava [1] established the following sufficient conditions for the absolute convergence of the defining integral (1.1) for the $\bar{H}$ - function :

$$\Omega \equiv \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} |A_j \alpha_j| - \sum_{j=M+1}^{Q} |B_j \beta_j| - \sum_{j=N+1}^{P} A_j > 0 \qquad \ldots (1.3)$$

and

$$|\arg(z)| < \pi\, \Omega/2., \qquad \ldots (1.4)$$

The behavior of the $\bar{H}$-function for small values of $|z|$ follows easily from a result given by Rathie [6, p.306, Eq.(6.9)], we have

$$\bar{H}_{P,Q}^{M,N}[z] = O(\,|z|^{\alpha}), \quad \alpha = \min_{1\leq j\leq M}\,[\mathrm{Re}(b_j/B_j)], \quad |\,z\,| \to 0. \qquad \ldots (1.5)$$

The following integral is required to establish our main results:

$$\int_0^t x^{\rho-1}(c+bx)^{-\lambda} e^{Rx/2L} \sin\{\omega(t-x)\}\, \bar{H}_{P,Q}^{M,N}\left[zx^{u}(c+bx)^{-\varepsilon} \left| \begin{matrix} (a_j, A_j;\alpha_j)_{1,N}, (a_j, A_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, B_j;\beta_j)_{M+1,Q} \end{matrix} \right. \right] dx$$

$$= t^{\rho+1} c^{-\lambda} \sum_{r,k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} t^{2k+m} (-b/c)^m (Rt/2L)^r}{r!\,m!}$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[zt^{u} c^{-\varepsilon} \left| \begin{matrix} (1-\lambda-m,\varepsilon;1), (1-\rho-r-m,u;1), (a_j, A_j;\alpha_j)_{1,N}, (a_j, A_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, B_j;\beta_j)_{M+1,Q}, (1-\lambda,\varepsilon;1), (-1-2k-\rho-r-m,u;1) \end{matrix} \right. \right]. \qquad \ldots (1.6)$$

Throughout this paper $\bar{H}_{P,Q}^{M,N}[z]$ stands for the well-known $\bar{H}$-function. The symbol $[r/2]$ stands for the greatest integer in r/2.

The condition of validity of the integral is as follows:

(i) $\operatorname{Re}(\lambda)>0, \quad \min(u,\varepsilon)>0, \quad t>0,$

(ii) $|bt/c|<1, \; A>0,$

(iii) $|\arg z|<\Omega\pi/2$,

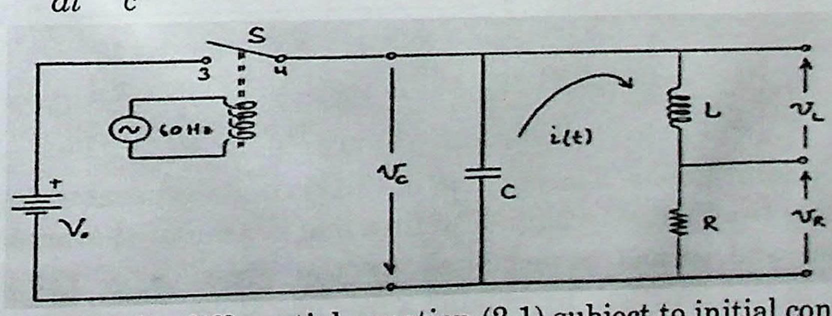
(iv) $\operatorname{Re}(\rho)+u\min\limits_{1\le j\le M}\{\operatorname{Re}(b_j/B_j)\}>0$ and the series on the right-hand side converges absolutely, it being understood that

$$\Omega\equiv\sum_{j=1}^{M}B_j+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|B_j\beta_j\right|-\sum_{j=N+1}^{P}A_j \;>0.$$

Proof. Expand the exponential function, sine function and we express the $\bar{H}$-function in terms of the contour integral (1.1), interchange the order of integration, which is permissible due to absolute convergence of the integrals involved in the process, and evaluate the integral by means of the integral due to Erdélyi [Tables of Integral Transforms, Vol.1, p. 310, (24)]. Expand the generalized hypergeometric function in series form and recollect the terms for $\bar{H}$-function to obtain the desired result (1.6).

2. Main Results. If we consider an electric circuit consisting of resistance R, an inductance L, a condenser of capacity C and a source of electromotive force $E_0P(t)$, where E_0 is a constant and $P(t)$ is known function of time t, the charge $q(t)$ on the plates of condenser at any time t, satisfies the following second order differential equation

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E_0P(t) \qquad \text{...(2.1)}$$



The solution of the differential equation (2.1) subject to initial conditions $q=Q, i=\dfrac{dq}{dt}=I$ when $t=0$, is the standard result [7] and is given below

$$q(t)=J(t)+\frac{E_0}{\omega L}e^{-Rt/2L}\int_0^t P(\eta)e^{R\eta/2L}\sin\{\omega(t-\eta)\}d\eta, \qquad \dots(2.2)$$

where, for convenience,

$$J(t)=e^{-Rt/2L}[Q\cos\omega t+(I_1/\omega)Sin\omega t], \qquad \dots(2.3)$$

$$I_1=I+RQ/2L \text{ and } (1/LC)-(R^2/4L^2)=\omega^2>0. \qquad \dots(2.4)$$

Now we find out the charge $q(t)$ when $P(t)$ is taken in terms of the Inayat Hussain $\bar{H}$-function.

We shall find the solution of (2.1), when $P(t)$ is taken in terms of $\bar{H}$-function. Let

$$P(t)=t^{\rho-1}(c+bt)^{-\lambda}.\bar{H}^{M,N}_{P,Q}\left[zt^{u}(c+bt)^{-\varepsilon}\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,N}, & (a_j,A_j)_{N+1,P}\\ (b_j,B_j)_{1,M}, & (b_j,B_j;\beta_j)_{M+1,Q}\end{matrix}\right.\right]. \qquad \dots(2.5)$$

Putting the above value of $P(t)$ in (2.2) and evaluating the integral with the help of (1.6), we find that the value of the charge $q(t)$ is given by

$$q(t)=J(t)+\frac{E_0}{\omega L}e^{-Rt/2L}F_1(r,k,m,t), \qquad \dots(2.6)$$

where $J(t)$ and $F_1(r,k,m,t)$stands for the quantities as given by (2.3) and (1.6), respectively and the condition as mentioned after (1.6) are satisfied.

The value of the current $\frac{dq}{dt}$ can also be obtained from (2.6), by differentiating the series on its right hand side term by term with respect to t. The process of term by term differentiation is assumed to be justified as the $\bar{H}$-function being analytic function and the resulting series of $\bar{H}$-function obtained in this case will be uniform convergent in any arbitrary domain $0\le t\le a$.

A special case of the solution (2.6), which is of practical interest, follows easily by putting $R=0$; thus we arrive at the following solution

$$q(t)=Q\cos\omega t+(I/\omega)\sin\omega t+c^{-\lambda}\sum_{k,m=0}^{\infty}\frac{(-1)^k(\omega)^{2k}(t)^{2k+m}(-b/c)^m E_0 t^{\rho+1}}{m!L}$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[zt^{u}c^{-\varepsilon}\left|\begin{matrix}(1-\lambda-m,\varepsilon;1),(1-\rho-m,u;1),(a_j,A_j;\alpha_j)_{1,N},(a_j,A_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,B_j;\beta_j)_{M+1,Q},(1-\lambda,\varepsilon;1),(-1-2k-\rho-m,u;1)\end{matrix}\right.\right]. \qquad (2.7)$$

3. Special Cases. The solution of (2.1) is quite general in nature as these possess two fold generality. The first one is general character of the $\bar{H}$-function and second is exhibited by the presence of the general arguments in this function. By making a free use of results which given by time to time many mathematicians, our solutions can be suitably applied to a remarkably wide variety of useful functions (or product of such functions)

that occur frequently in the problems of mathematical physics and engineering.

(i) If we take $M=1$, $N=2$, $P=2$, $Q=2$ in (1.1), the $\bar{H}$-function reduces to the generalized Riemann zeta function, shown by Buschman and Srivastava [1, p. 4708], as

$$\phi(z,q,\eta)=\sum_{k=0}^{\infty}\frac{z^k}{(\eta+k)^q}=\bar{H}_{2,2}^{1,2}\left[-z\left|\begin{matrix}(0,1;1),(1-\eta,1;q)\\(0,1),(-\eta,1;q)\end{matrix}\right.\right]. \quad ...(3.1)$$

Therefore, if electromotive force $E_0P(t)$ in (2.1) is taken in terms of the Riemann zeta function, as

$$E_0P(t)=t^{\rho-1}(c+bt)^{-\lambda}\phi(zt^u(c+bt)^{-\varepsilon},q,\eta)$$

$$=t^{\rho-1}(c+bt)^{-\lambda}\bar{H}_{2,2}^{1,2}\left[-zt^u(c+bt)^{-\varepsilon}\left|\begin{matrix}(0,1;1),(1-\eta,1;q)\\(0,1),(-\eta,1;q)\end{matrix}\right.\right]. \quad ...(3.2)$$

Charge at any time t is given by $q(t)=J(t)+\dfrac{E_0}{\omega L}e^{-Rt/2L}F_1(r,k,m,t)$, where

$$F_1(r,k,m,t)=t^{\rho+1}c^{-\lambda}\sum_{r,k,m=0}^{\infty}\frac{(-1)^k(\omega)^{2k+1}t^{2k+m}(-b/c)^m(Rt/2L)^r}{r!m!}$$

$$\bar{H}_{4,4}^{1,4}\left[-zt^uc^{-\varepsilon}\left|\begin{matrix}(1-\lambda-m,\varepsilon;1),(1-\rho-r-m,u;1),(0,1;1),(1-\eta,1;q)\\(0,1),(-\eta,1;q),(1-\lambda,\varepsilon;1),(-1-2k-\rho-r-m,u;1)\end{matrix}\right.\right]. \quad ...(3.3)$$

(ii) If we take the values of all $\alpha's=\beta's=1$ in (2.7), we find the solution of (2.1) for $R=0$, in terms of the results found out in [2].

(iii) If we place the values of all $A's=B's=\alpha's=\beta's=1$ in (2.7), and apply the well-known Gamma multiplication formula, we find the solution of (2.1) for $R=0$ in terms of the result found out in [3].

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Statement of ownership and particulars about the journal

JÑĀNĀBHA

1. **Place of Publication**	D.V. Postgraduate College Orai-285001, U.P., India
2. **Periodicity of Publication**	Annual
3. **Printer's Name** **Nationality** **Address**	Mr. Dheeraj Gupta Indian Customer Gallery, Orai Orai-285001, U.P., India Mobile : 9450296634
4. **Publisher's Name**	*Dr. R.C. Singh Chandel* For **Vijñāna Parishad of India**
Nationality **Address**	Indian D.V. Postgraduate College Orai-285001, U.P., India
5. **Editor's Name** **Nationality**	Dr. R.C. Singh Chandel Indian
Address	D.V. Postgraduate College Orai- 285001, U.P., India
6. **Name and address of the individuals who own the journal and partners of share holders holding more than one percent of the total capital**	**Vijñāna Parishad of India** Address : D.V. Postgraduate College Orai-285001, U.P., India

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